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Mathematical Reviews

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ANALYSIS

*Frank, Philipp und v. Mises, Richard. **Die Differential- und Integralgleichungen der Mechanik und Physik.** Mary S. Rosenberg, New York, 1943. 2200 pp. $27.50.
Photo-reprint of the second (latest) revised and enlarged edition [Vieweg, Braunschweig, 1930 and 1935].

Bellman, Richard. An integral inequality. Duke Math. J. 10, 547–550 (1943). [MF 8977]
This paper gives a new treatment of certain related inequalities previously studied by Carlson [Ark. Mat. Astr. Fys. 25B, 1–5 (1934)], Hardy [J. London Math. Soc. 11, 167–170 (1936)], Gabriel [J. London Math. Soc. 12, 130–132 (1937)] and Caton [Duke Math. J. 6, 442–461 (1940); these Rev. 2, 75]. *F. W. Perkins* (Hanover, N. H.).

Hornich, Hans. Eine Ungleichung für Vektorlängen. Math. Z. 48, 268–274 (1942). [MF 8553]
If $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_m$ are $m+n$ vectors in R_k for which

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j,$$

if $m \geqq 1$ and if c is a unit vector in R_k, then

$$\sum_{i=1}^{n} (|a_i+c| - |a_i|) \geqq \sum_{j=1}^{m} (|b_j+c| - |b_j|) + n + m - 2.$$

The proof involves analytic methods, an algebraic proof being given only for the case $m=1$, $n=2$. *R. P. Agnew.*

Krabill, David M. On extension of Wronskian matrices. Bull. Amer. Math. Soc. 49, 593–601 (1943). [MF 8860]
The matrix $M_s(u_1, \cdots, u_n) = \|u_i^{(j)}\|$ $(i=1, \cdots, n;$ $j=0, 1, \cdots, s)$, where $u_i^{(j)}$ is the jth derivative of $u_i(x)$, is called the Wronskian matrix of order s of $u_1, \cdots, u_n$. The author shows that, if $u_1(x), \cdots, u_n(x)$ are functions of class $C^{(t)}(t \geqq n)$ on $a \leqq x \leqq b$ and their Wronskian matrix of order s $(n \leqq s \leqq t)$ has rank n at every point of $a \leqq x \leqq b$, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_s(u_1, \cdots, u_{n+1})$ has rank $n+1$ on $a \leqq x \leqq b$. By the use of this result he shows that, if $u_1(x), \cdots, u_n(x)$ are of class $C^{(t)}(t \geqq n)$ on $a \leqq x \leqq b$ and $M_s(u_1, \cdots, u_n)$ $(s<t)$ has rank n on $a \leqq x \leqq b$, then the u's are linearly independent solutions of a homogeneous linear differential equation of order $s+1$ with leading coefficient unity. The paper employs a result due to Curtiss to the effect that, if $M_t(u_1, \cdots, u_n)$ $(t \geqq n)$ has rank n on $a \leqq x \leqq b$, the determinant of the matrix $M_{n-1}(u_1, \cdots, u_n)$ has at most isolated zeros. *M. R. Hestenes.*

Rochford, Mary de Pazzi. Integrability conditions without differentiability assumptions. Rep. Math. Colloquium (2) 4, 6–15 (1943). [MF 8958]
The author obtains several results on integrability conditions without differentiability assumptions, similar to those obtained by Menger, Fubini, Artin and Milgram in the past few years. One of the results obtained is the following theorem [theorem 6 of the paper]: let $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ be three functions defined and continuous in a parallelpiped D with faces parallel to the coordinate planes. With each rectifiable curve C contained in D associate the integral

$$J(C) = \int_C P\,dx + Q\,dy + R\,dz.$$

In order that in D the integral J be independent of the path it is necessary and sufficient that in D for each plane of the three families of planes $x=$const., $y=$const., $z=$const. the integral J be independent of the path. By combining this result with the results in the plane previously referred to, various conclusions can be drawn. *W. T. Martin.*

Theory of Sets, Theory of Functions of Real Variables

Rey Pastor, Julio. The lemma of Pincherle and the lemma of Borel. Revista Union Mat. Argentina 9, 29–35 (1943). (Spanish) [MF 8884]
The author points out that a lemma of Pincherle, serving roughly the same purpose as the Borel covering theorem and published prior to it, is not correct without an additional condition. He shows where some of Pincherle's proofs in his book on analytic functions are consequently not valid and proposes a modified form of the lemma, which is essentially Satz IV, p. 100 of Alexandroff-Hopf's Topologie, Springer, Berlin, 1935. The paper concludes with some remarks on nomenclature. *J. V. Wehausen* (Columbia, Mo.).

Best, E. On sets of fractional dimensions. III. Proc. London Math. Soc. (2) 47, 436–454 (1942). [MF 8270]
[For parts I and II see Proc. Cambridge Philos. Soc. 36, 152–159 (1940); 37, 127–133 (1941); these Rev. 1, 302; 3, 75.] The author considers the set $E_{r,s}$ of all numbers in (0, 1) which, expressed as decimals in the scale of r $(r>2)$, do not contain the digit s $(s<r)$. He shows that the Hausdorff-dimension of $E_{r,s}$ depends only on r and not on s. He also determines the measure of $E_{r,s}$ in some particular cases. *W. Hurewicz* (Providence, R. I.).

Albuquerque, J. La notion de mesure de Carathéodory. Portugaliae Math. 3, 258–262 (1942). [MF 8517]
Given a space with a closure operation φ and a real valued, nonnegative, nondecreasing, convex measure function Γ the author proves that a necessary and sufficient condition for each Borel set to be measurable is that $\Gamma(X+Y) \geqq \Gamma(X) + \Gamma(Y)$ whenever X and Y are sets of the space for which $X\varphi(Y)=0$. *J. F. Randolph* (Ithaca, N. Y.).

Adams, C. Raymond and Morse, Anthony P. On approximating certain integrals by sums. Trans. Amer. Math. Soc. 53, 363–426 (1943). [MF 8416]

Let E be a measurable linear set (bounded or unbounded), let f be a function Lebesgue integrable on E ($f \in L(E)$), let k be a fixed real number not less than 1 and let A_n and B_n be sequences (finite or infinite) of measurable subsets of E satisfying the conditions $A_n \subset B_n$, $0 < \text{measure } B_n = |B_n| \leqq k|A_n|$, $\text{diam}(B_n) \leqq \delta$ $(n=1, 2, 3, \cdots)$; $|B_m \cdot B_n| = 0$ for $m \neq n$, and $\sum B_n = E$. In a recent paper [Bull. Amer. Math. Soc. 45, 442–447 (1939)] the authors showed that

$$\int_E f(x)dx = \lim_{\delta \to 0} \sum \left[\int_{A_n} f(x)dx / |A_n| \right] |B_n|.$$

Upon letting, for the moment, E be an interval $[a, b]$, $f \in L^p$ on $[a, b]$, $p \geqq 1$, and $A_n = B_n =$ an interval $(n=1, 2, 3, \cdots)$, a result of Banach [Fund. Math. 3, 133–181 (1922), especially p. 175] may be expressed as

$$\int_E |f(x)|^p dx = \lim_{\delta \to 0} \sum \left| \int_{B_n} f(x)dx / |B_n| \right|^p |B_n|.$$

The present paper started with the investigation of whether

$$\int_E |f(x)|^p dx = \lim_{\delta \to 1} \sum \left| \int_{A_n} f(x)dx / |A_n| \right|^p |B_n|,$$

that is, upon letting $\varphi(y) = |y|^p$, whether

$$\int_E \varphi[f(x)]dx = \lim_{\delta \to 0} \sum \varphi \left[\int_{A_n} f(x)dx / |A_n| \right] |B_n|,$$

but soon progressed into investigations wherein the specific definition of φ was dropped.

For B a measurable subset of E, $0 < |B| < \infty$, let $\mathfrak{M}_B(f) = \int_B f / |B|$. As B varies, let $\mathfrak{R}(f)$ be the set of values of $\mathfrak{M}_B(f)$. The subset of E on which f is not in $\mathfrak{R}(f)$ is of measure zero. Let φ be a function whose domain includes $\mathfrak{R}(f)$. For $0 < \delta \leqq \infty$ let F be an arbitrary partition of E into essentially disjoint measurable sets each of diameter less than δ, and denote the aggregate of all such "δ-partitions" by $\Gamma_\delta(E)$. Let

$$S_*(f, \varphi, E) = \lim_{\delta \to 0} \inf_{F \in \Gamma_\delta(E)} \sum_{B \in F} \varphi[\mathfrak{M}_B f]|B|,$$

$S^*(f, \varphi, E)$ the limit with inf replaced by sup, and when and only when $S_*(f, \varphi, E) = S^*(f, \varphi, E)$ let $S(f, \varphi, E)$ denote this common value. Also for $k \geqq 1$ let $S_*(f, \varphi, E, k)$, $S^*(f, \varphi, E, k)$ and $S(f, \varphi, E, k)$ be defined by replacing $\mathfrak{M}_B f$ in the above by $\mathfrak{M}_{AB} f$, where A is a measurable set variable with F and for each F variable within the restriction (1) $|B| \leqq k|AB|$ for $B \in F$.

The main purpose of the paper is to obtain conditions on f and φ that insure the (finite) existence of $\int_E \varphi[f(x)]dx$ and $S(f, \varphi, E, k)$ and the equality

$$(2) \qquad \int_E \varphi[f(x)]dx = S(f, \varphi, E, k).$$

Theorems whose conclusions contain those of the following theorem are given. If f is integrable over each bounded subset of E and φ is continuous on $\mathfrak{R}(b)$, then (a) the condition $-\infty < S_*(f, \varphi, E) \leqq S^*(f, \varphi, E) < \infty$ is sufficient (as well as obviously necessary) for (2); (b) the conditions $|\varphi(y)| \leqq \psi(y)$ for $y \in \mathfrak{R}(f)$, ψ convex on $\mathfrak{R}(f)$, and $\int_E \psi[f(x)]dx < \infty$ are sufficient for (2). The condition of continuity on φ may not be dispensed with, but in (b) the condition of convexity of ψ is not necessary as shown by examples. The question as to whether (1) may be replaced by (3) $|AB| > 0$, $B \in F \in \Gamma_\delta(E)$, is also discussed. For example, it is shown that the conditions f Riemann integrable on $E = [a, b]$ and φ continuous on the closure of $\mathfrak{R}(f)$ are sufficient for (2) even with (1) replaced by (2).

The early part of the paper contains many results about sets and functions of sets in a generality greater than is necessary for the applications made of them. The last part is concerned with generalizations of the notions of continuity, absolute continuity and total variation that are relevant to the earlier work. The authors raise a question to which they do not know the answer. *J. F. Randolph.*

Price, G. Baley. Cauchy-Stieltjes and Riemann-Stieltjes integrals. Bull. Amer. Math. Soc. 49, 625–630 (1943). [MF 8865]

In the approximating sums

$$\sum_{i=1}^{n} f(\xi_i)[g(x_i) - g(x_{i-1})]$$

for the Riemann-Stieltjes integral $\int_a^b f dg$ the points ξ_i range over $x_{i-1} \leqq \xi_i \leqq x_i$. In this note the following modifications are considered: (a) Cauchy left: $\xi_i = x_{i-1}$, (b) Cauchy right: $\xi_i = x_i$, (c) right modified: $x_{i-1} \leqq \xi_i < x_i$, limits being either in the sense of refinement of subdivision, or as the norm (or maximum length of subinterval) of the subdivision approaches zero. By properly overlapping discontinuities of f and g, examples are given to show that the various definitions are not equivalent. Extending the result of Gillespie [Ann. of Math. (2) 17, 61–63 (1915)], it is shown that the existence of the Cauchy-Stieltjes integral with respect to a monotonic function g implies the existence of the corresponding Riemann-Stieltjes integral if f and g have no common discontinuities (on the same side) for the norm (refinement) integral. Similar results hold for the modified integral in the refinement case.

T. H. Hildebrandt (Ann Arbor, Mich.).

Helsel, R. G. and Young, P. M. Characterization theorems for integral means. Duke Math. J. 10, 259–269 (1943). [MF 8466]

The integral mean, defined on I_h: $h \leqq x \leqq 1-h$, of $f(x)$, defined and summable on I: $0 \leqq x \leqq 1$, is

$$f_h(x) = (1/2h) \int_{-h}^{h} f(x+\xi)d\xi,$$

h satisfying $0 < h < \frac{1}{2}$; the h-k-integral mean, defined on R_{hk}: $h \leqq x \leqq 1-h$, $k \leqq y \leqq 1-k$, of $f(x, y)$, defined and summable on S: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, is

$$f_h^k(x, y) = (1/4hk) \int_{-h}^{h} \int_{-k}^{k} f(x+\xi, y+\eta)d\xi d\eta,$$

where $0 < h, k < \frac{1}{2}$. Characterizations are obtained for these integral means (and some others) of functions of class C^n (possessing continuous derivatives up to the nth order inclusive, $n = 0, 1, 2, \cdots$) or L^p (measurable and of summable absolute value of their pth power, $p \geqq 1$). The main results are the following. (1) A necessary and sufficient condition that $\phi(x)$ be identical with $f_h(x)$ for $f(x)$ of class (a) C^n, (b) L^p on I is that it be (a) of class C^{n+1}, (b) absolutely continuous and possess a derivative of class L^p. (2) A necessary and sufficient condition that $\phi(x, y)$ be

identical with $f_h^k(x, y)$ for $f(x, y)$ of class (a) C^n, (b) L^p on S is that the three conditions hold: (α) ϕ is absolutely continuous in x for every y and in y for every x; (β) $\phi_x(x, y)$ ($\phi_y(x, y)$) is of class (a) C^n, (b) L^p and is absolutely continuous in y (x) for (a) every x (y), (b) almost every x (y); (γ) $\phi_{xy}(x, y)$ ($\phi_{yx}(x, y)$) is of class (a) C^n, (b) L^p. The various proofs are similar in character. Theorem 2 depends on an extension of the theorem of Zygmund that, for $f(x, y)$ summable on S, $F_{xy}=f=F_{yx}$ almost everywhere on S, where

$$F=\int_0^x\int_0^y f(\xi, \eta)d\xi d\eta.$$

H. Blumberg (Columbus, Ohio).

Huskey, Harry D. Contributions to the problem of Geöcze. Duke Math. J. **10**, 249–257 (1943). [MF 8465]

Let S be a continuous surface defined by $z=f(x, y)$, where $f(x, y)$ is defined and continuous on the closed unit square I. For a surface of this form the Lebesgue area $A[S]$ is defined to be inf lim inf $A[P_n]$, where P_n: $z=p_n(x, y)$, $n=1, 2, \cdots$, is a sequence of polyhedra defined over I, but not necessarily inscribed in S, such that $p_n(x, y)\rightarrow f(x, y)$ uniformly on I and the infimum is taken with respect to all such sequences of polyhedra. Let $A[P_n]$ denote the area of P_n in the elementary sense. Let $A^*[S]$ designate the quantity obtained by requiring that the polyhedra used in the definition of $A[S]$ be inscribed in S. The problem of determining whether $A^*[S]=A[S]$ is known as the problem of Geöcze since Geöcze was the first to make fundamental studies of it. Geöcze himself gave conditions on $f(x, y)$ under which $A^*[S]=A[S]$; recently these conditions have been greatly weakened by Radó [Amer. J. Math. **65**, 361–381 (1943); these Rev. **4**, 270; complete references to the work of Geöcze are given in this paper]. Geöcze stated that $A^*[S]=A[S]$ if $f(x, y)$ is merely continuous, but no proof is known. In the paper under review, Huskey proves the following theorem. If $f(x, y)$ is absolutely continuous in the sense of Tonelli, then $A^*[S]=A[S]$. The conditions in this theorem are much stronger than those obtained by Radó; interest in the results arises from the fact that the theorem is proved by two simple and direct methods, both of which are entirely different from those used in the study of the more general cases of the problem of Geöcze. The first proof is based on approximation to $f(x, y)$ by the integral mean function $f_h(x, y)$, where

$$f_h(x, y)=(1/h^2)\int_0^h\int_0^h f(x+\alpha, y+\beta)d\alpha d\beta.$$

The vertices of polyhedral surfaces inscribed in the given surface S and in the integral mean surface S_h are obtained by projecting, parallel to the z-axis, onto S and S_h the vertices of a triangular subdivision of the unit square I. By using known properties of $f_h(x, y)$, standard inequalities and some little known results on the approximation of Lebesgue integrals by means of Riemann sums first established by Geöcze [Bull. Soc. Math. France **39**, 256–295 (1911)] and later extended by Hahn [Akad. Wiss. Wien., S.-B. IIa **123**, 713–743 (1914)], it is shown that an integral mean function $f_h(x, y)$ and polyhedral surfaces inscribed in S and S_h can be chosen in such a way that $A[S]$ and $A[S_h]$, $A[S_h]$ and the area of the polyhedral surface inscribed in S_h, and the areas of the polyhedral surfaces inscribed in S and S_h each differ by an arbitrarily small quantity. It follows that $A^*[S]=A[S]$. The second proof is based on some recent results concerning convergence in area obtained by Radó and Reichelderfer [Duke Math. J. **9**, 527–565 (1942); these Rev. **4**, 75]. *G. B. Price* (Lawrence, Kan.).

Nöbeling, Georg. Über den Flächeninhalt dehnungsbeschränkter Flächen. Math. Z. **48**, 747–771 (1943). [MF 9012]

Let M, contained in Euclidean n-space R_n, be the image of an analytic set $\bar{M}\subset R_k$ under a mapping f that satisfies a Lipshitz condition: $\overline{f(p), f(q)}\leqq c\overline{p, q}$. It is shown that for such a set M the k-dimensional measures of Janzen, Carathéodory, Gross, Hausdorff, Favard, Kolmogoroff and Gillespie all have the same value $\mu(M)$, and in case the mapping is 1 : 1 this measure is given also by the classical integral formula. When the set $\bar{M}$ is the unit cube, the k-dimensional surface measures of Lebesgue, Peano and Radó are all equal to the classical integral, even when the mapping is not 1 : 1. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Fourier Series and Generalizations, Integral Transforms

Salem, R. Sets of uniqueness and sets of multiplicity. Trans. Amer. Math. Soc. **54**, 218–228 (1943). [MF 9103]

Let $0<\xi<\frac{1}{2}$ be a fixed number and let P denote the perfect and nondense set constructed as follows. The interval $0\leqq\theta\leqq 2\pi$ is divided into three parts, of lengths proportional to ξ, $1-2\xi$, ξ, respectively, and the interior of the middle part is removed. With each of the two remaining intervals we proceed similarly, and so on. What is finally left is the set P. It is of measure 0, and the problem is to determine the values of ξ for which P is a set of uniqueness for trigonometric series.

Let $\theta=1/\xi>2$. For the case of rational ξ the problem has been solved: the set P is one of uniqueness if and only if θ is an integer [cf. N. Bary, Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 699–722 (1937); R. Kershner, Amer. J. Math. **58**, 450–452 (1936)]. In particular, Cantor's well-known ternary set ($\theta=3$) is a set of uniqueness, a result known for a long time [Rajchman, Fund. Math. **3**, 287–302 (1922)]. The present paper presents the solution of the problem for general ξ: a necessary and sufficient condition that P be a set of uniqueness is that θ be an algebraic integer all of whose conjugates are inside the circle $|z|=1$. Basic for the (very interesting) argument are certain results of Pisot [Ann. Scuola Norm. Super. Pisa (2) **7**, 205–248 (1938)] and Vijayaraghavan [Proc. Cambridge Philos. Soc. **37**, 349–357 (1941); these Rev. **3**, 274]. It also turns out that, if P is a set of uniqueness, then it is a sum of a finite number of sets of type H, first considered by Rajchman [loc. cit.]. By choosing appropriate ξ's, we obtain sets of uniqueness whose Hausdorff dimension is as close to 1 as we wish.

A. Zygmund (South Hadley, Mass.).

Hornich, Hans. Über gewisse trigonometrische Integrale. I. Math. Z. **48**, 785–791 (1943). [MF 9015]

The author gives an elementary discussion of the cosine transforms of the functions

$$\prod_{i=1}^{k}\frac{\sin a_i x}{x},\quad \prod_{i=1}^{k}\frac{\sin a_i x}{a_i x}$$

for finite k and for $k\to\infty$. For finite k he gives an easy rule for computation by induction on k. His statements for $k\to\infty$ are covered by available theorems on positive definite functions. *S. Bochner* (Princeton, N. J.).

Loo, Ching Tsün. Note on strong summability of Fourier series. Acad. Sinica Science Record 1, 78–83 (1942). [MF 8842]

Let $s_n(x)$ be the partial sum of order n of the Fourier series of a function $f(x)\in L$. Hardy and Littlewood have proved [Fund. Math. 25, 162–189 (1935)] that at every point x such that

$$\int_0^h |f(x+t)+f(x-t)-2f(x)|\,dt=o(h) \tag{1}$$

one has

$$\sum_{\nu=0}^{n}(s_\nu-f)^2=o(n\log n).$$

The purpose of this paper is to prove that under the condition (1) one has also

$$\sum_{\nu=0}^{n}(s_{\nu k}-f)^2=o(n\log n)$$

for every positive integer k. *R. Salem.*

Bellman, Richard. Convergence of non-harmonic Fourier series. Duke Math. J. 10, 551–552 (1943). [MF 8978]

The author generalizes a theorem of the reviewer [Duke Math. J. 8, 541–545 (1941); these Rev. 3, 107] and proves that the series $\sum_1^\infty a_k e^{i\lambda_k x}$ converges almost everywhere, provided that, for some p $(0<p<2)$, $\sum|a_k|^p<\infty$ and a $b>0$ exists such that $\lambda_{k+1}-\lambda_k\geqq b$ for sufficiently large k. The proof employs Zygmund's generalization of Menchoff's theorem [Proc. Cambridge Math. Soc. 32, 321–327 (1936)] and a device similar to the one used by the reviewer [loc. cit., pp. 541–542]. *M. Kac* (Ithaca, N. Y.).

Kac, M. Convergence of certain gap series. Ann. of Math. (2) 44, 411–415 (1943). [MF 8871]

The author investigates the convergence properties of series of the form $\sum_{n=1}^\infty c_k\varphi(n_kx)$, where $\{n_k\}$ is a gap sequence of integers $(n_{k+1}/n_k>q>1)$ and $\varphi(x)$ a complex valued periodic function satisfying Hölder's condition and having a zero average. He shows that, if $\sum_{n=1}^\infty|c_k|^2<\infty$, the series of functions converges in the mean with exponent 2 over every finite interval to a periodic function $f(x)$, and that the series converges almost everywhere to $f(x)$. He points out that earlier results along this line by himself and by Izumi and Kawata required the n_k to be powers of 2, whereas nothing is specified as to the arithmetic nature of the present gap sequence. He also shows that, unlike the earlier results, the present hypotheses do not necessarily make the series diverge almost everywhere (or, in fact anywhere) when $\sum_{n=1}^\infty|c_k|^2$ diverges. *R. H. Cameron* (Cambridge, Mass.).

Wintner, Aurel. The behavior of Euler's product on the boundary of convergence. Duke Math. J. 10, 429–440 (1943). [MF 8964]

The almost periodicity (B^2) of a function $g(t)$ implies, in particular, the existence of the mean value $M(|g(t)|^2)$. Conversely, for boundary values of analytic almost periodic functions (Bohr), the existence of the mean value is in many cases strongly suggestive (even in matters of proof) of full almost periodicity (B^2). In a previous paper [Duke Math. J. 2, 443–446 (1936)] the author discussed almost periodicity (B^2) of $1/\zeta(1+it)$; however, the existence of the mean value had been anticipated by Landau and Schnee. In the present paper the author extends his treatment to the functions $\zeta'/\zeta(1+it)$ and $\log\zeta(1+it)$ for which the existence of the mean value had not been known before. *S. Bochner* (Princeton, N. J.).

Erdélyi, A. Inversion formulae for the Laplace transformation. Philos. Mag. (7) 34, 533–537 (1943). [MF 9133]

A new method of obtaining inversion formulas for

$$g(s)=\int_0^\infty e^{-st}f(t)\,dt$$

is announced, without proofs. For example, put $e^{-\frac{1}{2}t}=\sin\frac{1}{2}\theta$ and expand $f(t)\cot\frac{1}{2}\theta\sin^{2a}\frac{1}{2}\theta$ in a cosine series $\frac{1}{2}a_0+\sum_1^\infty a_n\cos n\theta$. Then $a_n=(2/\pi)\sum_{m=0}^\infty c_m g(a+m)$, where the c_m are the coefficients of the expansion

$$\cos n\theta=\sum_{m=0}^{n}c_m\sin^{2m}\tfrac{1}{2}\theta,$$

and hence

$$f(t)=e^{at}\tan\tfrac{1}{2}\theta\left\{\tfrac{1}{2}a_0+\sum_1^\infty a_n\cos n\theta\right\},\quad \sin\tfrac{1}{2}\theta=e^{-\frac{1}{2}t}.$$

Another formula is obtained by putting $1-2e^{-t}=x$ and expanding $e^{(1-b)t}f(t)$ in a series of Legendre polynomials. Then $f(t)=e^{(b-1)t}\sum_0^\infty b_nP_n(x)$, where $b_n=(2m+1)\sum_{m=0}^n d_m g(b+m)$, and the d_m are the coefficients in $P_n(x)=\sum_{m=0}^n d_m e^{-mt}$. In both cases the coefficients a_n, b_n depend only on the values of $g(s)$ at points of an arithmetic progression. Substituting the inversion formulas into the definition of $g(s)$, one obtains interpolation formulas such as

$$g(s)=\sum_{n=0}^{\infty}{}'\,a_n\frac{(-1)^n2^{2a-2s}\pi\Gamma(2s-2a+1)}{\Gamma(s-a+n+1)\Gamma(s-a-n+1)}.$$

R. P. Boas, Jr. (Cambridge, Mass.).

Schaeffer, A. C. On the oscillation of differential transforms. III. Oscillations of the derivative of a function. Trans. Amer. Math. Soc. 54, 278–285 (1943). [MF 9105]

Proofs of results announced previously [Proc. Nat. Acad. Sci. U. S. A. 28, 62–64 (1942); these Rev. 3, 144]. (a) If $f(x)\in C^\infty$ in an open interval I and if no derivative of f changes sign more than p times in I (p fixed), then f is analytic in I [for $p=0$, the result had been proved by S. Bernstein, Leçons sur les propriétés extrémales . . . , Gauthier-Villars, Paris, 1926, pp. 196–197]. (b) If $f\in C^\infty$ in $(-\infty,+\infty)$, if the number of changes of sign of every $f^{(k)}(x)$ in any interval of length α does not exceed β (α and β independent of k) and if

$$\log|f(x)|=O(|x|) \tag{*}$$

for $x\to\pm\infty$, then f is an entire function of exponential type. (c) If f satisfies the conditions of (b) with $\alpha=\pi$ and if f is bounded over $(-\infty,+\infty)$, then

$$|f(z)|\leqq Me^{\tau|y|},\qquad z=x+iy.$$

(d) If $f\in C^\infty(-\infty,+\infty)$ satisfies (*) and if $f^{(k)}(x)$ changes sign no more than $O(\log^\gamma k)$ $(\gamma<1)$ times in any interval of fixed length α, then f is an entire function of order not exceeding 1. *A. Zygmund* (South Hadley, Mass.).

Bary, Nina. Sur la stabilité de certaines propriétés des systèmes orthogonaux. Rec. Math. [Mat. Sbornik] N.S. 12(54), 3–27 (1943). (French. Russian summary) [MF 8788]

This paper discusses the properties which are shared by *ON* sets in L^2 "near" each other in some suitable sense. Various degrees of nearness are defined as follows. Let $\{\varphi_n\}$, $\{\psi_n\}$ be *ON* sets defined on an interval (a, b), $\rho_n=\|\varphi_n-\psi_n\|$. The sets are near (proches) if $\rho_n\to 0$, near in the restricted sense (proches au sens restreint) if $\sum\rho_n<\infty$; if $\epsilon_n\to 0$ then $\{\psi_n\}$ is in the $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots\}$-neighborhood of $\{\varphi_n\}$ if $\rho_n\leqq\epsilon_n$.

Let P be a property of a system $\{\varphi_n\}$. Then P is (a) weakly stable, (b) stable or (c) strongly stable if (a) for each $\{\psi_n\}$ in a certain $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots\}$-neighborhood of $\{\varphi_n\}$, (b) for each $\{\psi_n\}$ near $\{\varphi_n\}$ in the restricted sense or (c) for each $\{\psi_n\}$ near $\{\varphi_n\}$, respectively, there exists a set of functions $\bar{\psi}_n$ with property P such that $\psi_n=\bar{\psi}_n$ almost everywhere (a.e.). The principal properties discussed are the following. (A) The convergence of $\sum a_n\phi_n(x)$ on a set of positive measure implies $a_n\to 0$. (B) The same for convergence a.e. (C) The Fourier series of any function in L^2 with respect to the set $\{\varphi_n\}$ converges a.e. The functions $\{\varphi_n\}$ are (D) uniformly bounded, (E) uniformly bounded and continuous, (F) continuous, (G) bounded individually. The following results are obtained: (A), (B) are strongly stable, (C) is stable. When the *ON* sets in question are also complete the following more delicate results hold: (D), (E) are stable, (F), (G) are weakly stable. It is proven that completeness is essential for the latter results. *H. Pollard.*

Braunschmidt, Otto. Über Interpolation. J. Reine Angew. Math. **185**, 14–55 (1943). [MF 8815]

This paper gives a unified treatment of the general problem of interpolation where a function $f(x)$ is to be approximately represented by a linear expression of the form

$$P_n(x)=A_0u_0(x)+A_1u_1(x)+\cdots+A_nu_n(x),$$

in which the basic functions $u_i(x)$ are given linearly independent functions. The treatment includes as particular cases the various methods of polynomial interpolation (Lagrange, Newton, Sterling, Everett, etc.), the method of least squares (Gauss, Gram, etc.), trigonometric interpolation (either by sums or integrals) and other special methods associated with the names of Cauchy, Tschebyscheff, Fischer-Hinnen, etc. The author's procedure is to express $P_n(x)$ in the form

$$P_n(x)=\sum_{k=0}^{m}K_n(t_k, x)f(t_k)$$

for the case of discrete points or

$$P_n(x)=\int_a^b K_n(t, x)f(t)dt$$

in the continuous case. The kernel $K_n(t, x)$ is expressed in the form

$$K_n(t, x)=\chi_0(t)\varphi_0(x)+\chi_1(t)\varphi_1(x)+\cdots+\chi_n(t)\varphi_n(x),$$

where the φ's are linear combinations of the u's, and the χ's and φ's form a biorthogonal system. The procedure is then applied to a number of the different types of interpolation mentioned above. *W. E. Milne.*

***Shohat, J. A. and Tamarkin, J. D. The Problem of Moments.** American Mathematical Society Mathematical Surveys, vol. II. American Mathematical Society, New York, 1943. xiv+140 pp. $2.25.

The problem of moments is the theory of the infinite system of integral equations $\mu_n=\int t^n d\psi(t)$ $(n=0, 1, 2, \cdots)$ under various hypotheses on the range of integration and the function $\psi(t)$. The authors are principally concerned with the Hamburger moment problem, where $\psi(t)$ is nondecreasing and the range of integration is $(-\infty, \infty)$. While a few results on this problem, such as the determinant condition for the existence of a solution and Carleman's sufficient condition for its uniqueness, have become widely known, no unified and detailed presentation has hitherto been available of the deeper theory of the moment problem, which furnishes quite precise information about its solutions. This state of affairs is perhaps due to the large amount of analytic machinery, belonging to various fields, which has been used in investigations of the problem. The authors have succeeded to a large extent, in the first two chapters of this book, in presenting a unified and self-contained account of the principal results. The third chapter gives an idea of the wealth of generalizations and special problems to which the moment problem has given rise; the theory of many of these is still far from complete. The last chapter is devoted to a discussion of the approximate integration formulae arising from the moment problem. There is an extensive bibliography, which goes considerably beyond the literature actually referred to.

The first chapter starts with a general criterion (nonnegativeness of a functional defined by the μ_n) for solubility of the multidimensional moment problem, where the integration is over k-dimensional space, t^n is replaced by a product of powers, and $\psi(t)$ is replaced by a k-dimensional distribution function with spectrum in a prescribed set. Special cases yield well-known conditions for the Hamburger problem $(\int_{-\infty}^{\infty})$, Stieltjes problem $(\int_0^\infty)$, Hausdorff problem $(\int_0^1)$, two-dimensional Hausdorff problem $(\int_0^1\int_0^1)$ and trigonometric moment problem (integral around the unit circle). Carleman's criterion $(\sum\mu_{2n}^{-1/(2n)}=\infty)$ for a Hamburger problem to be determined (that is, to have a unique solution) is derived and extended to the multidimensional case.

Chapter 2 discusses the Hamburger problem in detail by means of the integral $\int_{-\infty}^{\infty}(z-t)^{-1}d\psi(t)$, asymptotic series and quasi-orthogonal polynomials. The exposition follows R. Nevanlinna and M. Riesz. Various properties of the solution depend on a function $\rho(z)$ which can be defined in a number of different ways; a detailed investigation of $\rho(z)$ is included. The discussion is practically self-contained, except for some properties of the so-called extremal solutions in the indeterminate case; however, the authors give M. Riesz's theorem connecting these solutions with Parseval's formula for the orthogonal polynomials associated with a given sequence $\{\mu_n\}$. A number of necessary and sufficient conditions are given for a moment problem to be determined. In chapter 3, various modifications of the moment problem are discussed in detail, and others are mentioned. The following are some of the topics discussed: Markoff's problem of bounds for $\int_1^\infty f(t)d\psi(t)$ and $\int_a^1 f(t)d\psi(t)$, when $\int_{-1}^{1}t^n d\psi(t)=\mu_n$ $(n=0, 1, 2, \cdots, k)$; criteria for the Hamburger problem to have an absolutely continuous solution with bounded derivative; the Hausdorff moment problem as related to completely monotonic sequences and

functions; the Hausdorff problem when $\psi(t)$ is not required to be monotonic, but is restricted in other ways.

Chapter 4 is of quite a different character. A particular problem, that of the properties of the approximate quadrature formula based on the roots of quasi-orthogonal polynomials associated with a moment problem, is discussed in detail. The emphasis is on convergence criteria. Of particular interest in the light of recent investigations is the connection between convergence properties and functions for which a sequence of derivatives have a fixed sign.

R. P. Boas, Jr. (Cambridge, Mass.).

Differential Equations

Vicente Gonçalves, J. Sur la formule de Rodrigues. Portugaliae Math. 4, 52–64 (1943). [MF 9130]

The differential equation (I) $Ay''+By'+cy=0$, in which $A=a_0x^2+a_1x+a_2$, $B=b_0x+b_1$, has a particular solution given by

$$Y=Ae^{-\Phi}(d^n/dx^n)(e^{\Phi}A^{n-1})$$

(formula of Rodrigues), where $\Phi=\int BA^{-1}dx$ and n is a positive integer which is assumed to be a root of the equation $E(\xi)=a_0\xi(\xi-1)+b_0\xi+c=0$; Y is a polynomial in x of degree not exceeding n, but it might happen that $Y\equiv 0$, in which case $e^{\Phi}A^{n-1}$ is a polynomial of degree $q<n$, and the equation $E(\xi)=0$ then has the additional positive integral root $m=n-1-q$. In this case (I) has two polynomial solutions Y^* and Y^{**} of degrees m and n, respectively, which are determined from

$$Y^*=Ae^{-\Phi}(d^m/dx^m)(e^{\Phi}A^{m-1})$$

and

$$(d^{m+1}/dx^{m+1})Y^{**}=e^{-\Phi}A^{-m}.$$

Also $Y\equiv 0$ is a necessary and sufficient condition that (I) have two polynomial solutions. *D. Moskovitz.*

Camp, Chester C. A convergence proof involving an inseparable multiple contour integral. Amer. J. Math. 65, 216–220 (1943). [MF 8198]

This paper is a sequel to and depends upon an earlier paper by the same author [Amer. J. Math. 60, 447–452 (1938)]. Consider a system of differential equations

$$(1)\qquad X_j'+\left[\sum_{i=1}^{p}\lambda_i a_{ji}(x_j)\right]X_j=0,\quad j=1, 2, \cdots, p,$$

with the auxiliary conditions

$$(2)\quad X_j(a_{ij})+C_{j1}X_j(a_{2j})+\cdots+C_{j,k_j-1}X_j(a_{k_j,j})+C_{j,k_j}X_j(b_j)=0,$$

where $C_{j,k_j}\neq 0$; $j=1, \cdots, p$. In the earlier paper the author considered, among other things, the expansion of a fairly general function $f(x_1, \cdots, x_p)$ in terms of solutions of the system (1). He assumed that the coefficients of the parameters λ_i maintained this average value under every subinterval for the variable x_j; that is,

$$(3)\qquad \int_{a_{ij}}^{a_{i+1,j}} a_{jk}(x_j)dx_j/(a_{i+1,j}-a_{ij})=A_{jk},\quad i=1, 2, \cdots, k_j-1.$$

In the present paper he indicates how to remove this restriction. (The work is actually carried through for the case $p=2$. The author states that the results for the case $p>2$ can be extended in similar manner.) The proof uses the notion of the residue of a double integral in the space of two complex variables. Assuming that each $a_{ji}(x_j)$ is integrable and either positive or identically zero in the domain T: $a_j\leqq x_j\leqq b_j$; $j=1, 2$, and assuming that the determinant of the integrals of the $a_{ji}(x_j)$ over the respective intervals are different from zero, the author shows that a properly restricted function $f(x_1, x_2)$ has an expansion in a series of characteristic solutions. Results on convergence properties are obtained. In certain cases in which the characteristic places are not simple, it is found necessary to add a term to the expansion. *W. T. Martin.*

Makai, E. Zur Berechnung der Eigenfrequenzen inhomogener Saiten. Z. Angew. Math. Mech. 22, 167–168 (1942). [MF 8900]

The condition upon the coefficient $q(x)$ under which the characteristic values of the boundary problem

$$y''+\lambda q(x)y=0,\qquad y(0)=y(1)=0,$$

are given by the explicit formula

$$\lambda_n=\left(n\pi\Big/\left(\int_0^1\sqrt{q(u)}du\right)\right)^2+C$$

is known to be that

$$(1)\qquad 4qq''-5q'^2=Cq^3.$$

This paper gives an elementary explicit integration of this equation (1). *R. E. Langer* (Madison, Wis.).

Bilharz, Herbert. Über eine gesteuerte eindimensionale Bewegung. Z. Angew. Math. Mech. 22, 206–215 (1942). [MF 8918]

The paper gives all periodic solutions of the equation $\ddot{x}+a\dot{x}+bx+c\ \mathrm{sgn}(x+k\dot{x})=0$, where a, b, c, k are constants, $a, b, c>0$. The periodic solutions exist only if $k<0$. They are stable in the sense of Trefftz [Math. Ann. 95, 307–312 (1925)]. All other stable solutions converge towards the periodic ones as $t\to\infty$. The solutions are discussed in the phase space. *I. Opatowski* (Chicago, Ill.).

Bückner, Hans. Über eine Näherungslösung der gewöhnlichen linearen Differentialgleichung 1. Ordnung. Z. Angew. Math. Mech. 22, 143–152 (1942). [MF 8896]

Under certain conditions on $\lambda(x)$ an approximate solution of the linear differential equation

$$(1)\qquad y+\lambda y'=f(x)$$

is given by

$$(2)\qquad y=f(x+u),$$

the function $u(x)$ being defined by the equation $\lambda(x+u)+u=0$. While the general solution of (1) is easily written down, it does not bring out the effect of changing $f(x)$ on the solutions of (1) in the same light as the approximate solution (2). The author applies his approximating function $f(x+u)$ to several engineering problems. *F. G. Dressel.*

Catunda, Omar. On systems of total variational equations in more than one unknown functional. Anais Acad. Brasil. Ci. 14, 109–125 (1942). (Portuguese) [MF 7262]

The first published proofs of existence and uniqueness theorems for completely integrable total differential equations with functionals or functional transformations as unknowns were given by Michal and Elconin [Acta Math. 68, 71–107 (1937), in particular, pp. 95–96]. These theorems

were obtained as corollaries of more general existence and uniqueness theorems of total differential equations in Banach spaces. [For applications to differential geometry and continuous groups see Michal, Bull. Amer. Math. Soc. **45**, 529–563 (1939); these Rev. **1**, 29.] In this paper the author proves an existence and uniqueness theorem for completely integrable total differential equations with n unknown numerically valued (complex) functionals of r functions of several complex variables. He proves the theorem that there exists one and only one set of n solution functionals, regular in certain functional regions, of the r functions that takes on initially given numerical values for r arbitrarily assigned holomorphic functions under the hypothesis of complete integrability of the equations and the regularity in suitable regions of the given functionals of the equations. The notions of analyticity and regularity of functionals are those given by Fantappiè [Accad. Naz. Lincei. Mem. (6) **3**, 1–233 (1930); Atti Accad. Italia. Mem. **2**, 1–172 (1931)]. This is a theorem in the small [see the paper by Michal and Elconin] in that it does not furnish any information as to the dependence of the solution functionals on the $n+r$ initial parameters. *A. D. Michal* (Pasadena, Calif.).

Keldych, M. On Galerkin's method of solution of boundary problems. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 309–330 (1942). (Russian. English summary) [MF 8503]

In 1915 Galerkin introduced a method for the approximate solution of differential equations which, in the case of self-adjoint equations, reduces to the Ritz method. However, since it also applies to other cases, it represents a generalization of the Ritz method. The author proves that in several important cases the Galerkin approximation functions converge to the true solution. First he considers the ordinary differential equation

$$\mathbf{L}(y)\equiv p(x)y^{(2n)}(x)+\sum p_k(x,\lambda)y^{(k)}(x)=f(x),$$

where the $p_k(x,\lambda)$ are continuous functions of a real variable x and of a complex variable λ. They are defined for $0\leqq x\leqq 1$ and $\lambda\varepsilon D$. For fixed x they are supposed to be analytic functions of λ and regular in D. The boundary conditions are $y^{(k)}(0)=y^{(k)}(1)$, $k=0, 1, \cdots, n-1$. Let $\{\phi_\nu(x)\}$ be a system of functions satisfying the boundary conditions. The Galerkin method requires the determination of approximations

$$y^{(m)}(x)=\sum_{\nu=1}^{m} X_\nu^{(m)}\phi_\nu(x)$$

in such a way that

$$\int_0^1 [\mathbf{L}(y^{(m)})-f]\phi_\nu dx=0, \quad \nu=1, 2, \cdots, m.$$

Let $\lambda_k^{(m)}$, $y_k^{(m)}(x)$ and λ_k, $y_k(x)$ represent the characteristic values and the characteristic functions of the approximate Galerkin system and of the exact equation $\mathbf{L}(y)=f$, respectively. If the system $1, x, \cdots, x^{n-1}, \phi_1^{(n)}, \phi_2^{(n)}, \cdots$ is complete in L^2, then, as the author shows, $\lim_{m\to\infty}\lambda_\nu^{(m)}=\lambda_\nu$, $\lim_{m\to\infty}y_\nu^{(m)}(x)=y_\nu(x)$. If λ is not a characteristic value, then the Galerkin approximations converge to the corresponding solution of the equation $\mathbf{L}(u)=f$. For an equation of second order, the author considers the case of boundary conditions $y'(0)=h_0y(0)$, $y'(1)=h_1y(1)$, where the h_k are constants. If the system $\{\phi_\nu\}$ is complete in the sense indicated, then the Galerkin approximate solutions converge to the corresponding solution of the equation.

Next the author considers the equation

$$\mathbf{L}(u)=\sum_{\nu=1}^{n}\left\{\partial\left[\sum_{k=1}^{n}p_{\nu k}\partial u/\partial x_k\right]\Big/\partial x_\nu\right\}+\sum_{k=1}^{n}a_k[\partial u/\partial x_k]+bu=f$$

of elliptic type, where $a_k=a_k'+\lambda a_k''$, $b=b'+\lambda b''$ and u satisfies the equation $\mathbf{L}(u)=f$ in a domain D and vanishes on the boundary. Let $\{\phi_k(x_1, x_2, \cdots, x_n)\}$ be a system of functions which are twice continuously differentiable, satisfy the boundary conditions and are complete in the following sense. For every function $\phi(x_1, \cdots, x_n)$ which vanishes on the boundary and for which

$$\int_D |\text{grad }\phi|^2 dx_1dx_2\cdots dx_n<\infty,$$

there exists a sequence $\phi_m=\sum_{k=1}^{m}\alpha_k\phi_k$ such that

$$\lim_{m\to\infty}\int_D |\text{grad }(\phi/\phi_m)|^2 d\omega=0.$$

Then the Galerkin approximation functions $u^{(m)}=\sum y_k^{(m)}\phi_k$, which are obtained from

$$\int_D [\mathbf{L}(u^{(m)})-f]\phi_k d\omega=0, \quad k=1, 2, \cdots, m,$$

converge in the mean to the solution u, and the approximate characteristic values $\lambda_\nu^{(m)}$ converge to the characteristic values λ_ν of the equation. *S. Bergman.*

Reade, Maxwell. A theorem of Fédoroff and Binney. Duke Math. J. **10**, 537–538 (1943). [MF 8975]

The author proves the following theorem: if $u(x, y)$ and $v(x, y)$ are continuous in D: $x^2+y^2\leqq 1$, and if

$$(1)\quad \int_R u(x,y)dy-v(x,y)dx\geqq 0, \quad \int_R u(x,y)dx+v(x,y)dy=0$$

hold for each oriented rectangle R in D, then there exists a function $\Phi(x, y)$, subharmonic in D, such that $\Phi_x(x, y)=u(x, y)$ and $\Phi_y(x, y)=v(x, y)$; the mass function associated with $\Phi(x, y)$ is given by the nonnegative completely additive extension of the rectangle function

$$(3)\quad \mu_1(R)\equiv\int_R u(x,y)dy-V(x,y)dx=\int_R \Phi_x(x,y)dy-\Phi_y(x,y)dx.$$

The proof is simple and direct. The proof of the first portion depends upon a result on subharmonic functions due to J. H. Levin, while that of the second portion depends upon an extension theorem of P. Reichelderfer and L. Ringenberg. The theorem is similar to one due to V. S. Fédoroff and J. H. Binney on harmonic functions. In their result the equality is assumed in both relations in (1) and the conclusion is that $\Phi(x, y)$ is harmonic in D. The author states that an immediate proof of his results can be derived from the deep results obtained by Binney, but he gives instead the simple proof mentioned. *W. T. Martin.*

Reade, Maxwell. Some remarks on subharmonic functions. Duke Math. J. **10**, 531–536 (1943). [MF 8974]

If $m(f, x, y, r)$ is the perimeter mean of the function $f(x, y)$ over the circle of center (x, y) and radius r, and

$M(f, x, y, r)$ the corresponding area mean, the operators

$$P(f, x, y) = \overline{\lim_{r \to 0}} \frac{M(f, x, y, r) - f(x, y)}{\frac{1}{8}r^2},$$

$$B(f, x, y) = \overline{\lim_{r \to 0}} \frac{m(f, x, y, r) - f(x, y)}{\frac{1}{4}r^2}$$

have been considered by Privaloff and Blaschke, respectively, who showed that necessary and sufficient conditions that f be subharmonic are that $P \geqq 0$ or $B \geqq 0$. The author considers the corresponding operators where the means are computed over oriented equilateral triangles and obtains the corresponding sufficient conditions but not the necessary ones. If the mass distribution μ appearing in the Riesz decomposition of the subharmonic function f has a bounded density $D_s\mu(x, y)$, it is shown that

$$\lim_{r \to 0} \frac{A - f}{\frac{1}{8}r^2} = \lim_{r \to 0} \frac{a - f}{\frac{1}{4}r^2} = \lim_{r \to 0} \frac{a - A}{\frac{1}{8}r^2} = D_s\mu(x, y)$$

almost everywhere, where A, a are the perimeter and area means over the triangles used and r is their circumradius. This implies the necessity portion of the theorem stated above for this class of subharmonic functions. It is also shown that, wherever $D_s\mu > 0$, the triangular averages converge monotonely to f, a fact which is known not to be true in general. *J. W. Green* (Aberdeen, Md.).

Copson, E. T. Some applications of Marcel Riesz's integrals of fractional order. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 260–272 (1943). [MF 8255]

With

(a) $$\partial^2/\partial x_1^2 - \sum_2^4 \partial^2/\partial x_i^2,$$

M. Riesz [C. R. Congrès International des Mathématiciens, vol. 2, Oslo, 1936, pp. 44–45, 62–63] has associated the operator

(b) $$I^\alpha \cdot = K(\alpha) \int_D \cdot R^{\alpha - 4} dx_1 \cdots dx_4,$$

where

$$R = (x_1 - x_1')^2 - \sum_2^4 (x_i - x_i')^2$$

and D is the domain bounded by $R = 0$ and $x_1' = 0$. The operators I^α are the counterparts for (a) of the Riemann or Volterra fractional integrals arising from (c) d/dx. Thus, for instance, operational solutions of problems involving the operator (a) are possible using I^α. In the present paper such operational solutions are given for the nonhomogeneous wave and damped wave equations with vanishing Cauchy data on $x_1 = 0$. Moreover, the author furnishes compact proofs for statements made in Riesz's short notes so the final results have a rigorous foundation. An operator J^α similar to (b) is defined by Riesz for the Laplacian Δ and under suitable conditions (d) $\Delta J^2 u = -u$. The last part of the paper is devoted to a formal derivation of Fock's integral equation in momentum space for the hydrogen atom and a more complicated one for helium. In the course of the analysis, symbolic juggling, such as writing (e) $1/\Delta^{\frac{1}{2}} = J(-1)^{\frac{1}{2}}$ (cf. (d)), is used freely. An alternative Fourier integral argument lends confirmation to the use made of (e). *D. G. Bourgin* (Urbana, Ill.).

Soboleff, S. Quelques problèmes limites nouveaux pour les équations aux dérivées partielles du type hyperbolique. Rec. Math. [Mat. Sbornik] N.S. 11(53), 155–203 (1942). (Russian. French summary) [MF 8367]

The author studies (a) stability of s.'s (solutions) in a domain Ω under self-adjoint b.c.'s (boundary conditions); on the basis of (a) he treats (b) problem 1: to find a s. subject only to b.c.'s on the frontier S of Ω, the s. having at ∞ assigned asymptotic e. (evaluation) and (c) problem 2: to find a s. of

(1) $$\sum \frac{\partial^2 w}{\partial x_i^2} - \frac{\partial^2 w}{\partial t^2} = \psi$$

subject to a b.c. on a cone. He seeks a s. of

(2) $$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} + Cu = \rho \frac{\partial^2 u}{\partial t^2} + \rho F,$$

$\rho > 0$; $\sum A_{ij} p_i p_j$ positive definite, subject to

$$u|_{t=t_0} = u_0, \ (\partial u/\partial t)|_{t=t_0} = u_1$$

and to b.c.'s (3a) $u = 0$ or (3b) $\partial u/\partial v = 0$ (on S). The known e.'s of the

$$Y_{k,\beta}^{*2} = \sum \frac{\beta!}{\gamma_1! \cdots \gamma_n!} \int \cdots \int \rho \left(\frac{\partial^k u}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n} \partial t^{k-\beta}} \right)^2 d\Omega$$

(Courant, Friedrichs, Levy, etc.) are for a bounded interval of t, while the author needs e.'s for all t. Supposing that $Lu + \lambda\rho u = 0$, under (3), has characteristic values and functions $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{m-1} \leqq 0 < \lambda_m \leqq \cdots$, V_1, V_2, $\cdots$, while $F = \sum_1^n f_i(t) V_i + F_m$ (F_m orthogonal to V_1, $\cdots$, V_{m-1}), he finds a s. $u = \sum a_i(t) V_i + u_m$, where u_m has e. (4) $Y_{0,0} \leqq \lambda_m^{-\frac{1}{2}}[\alpha_1 + \int_{t_0}^{t} f_{00} dt_1]$ (similar e.'s for $Y_{1,0}$, $Y_{1,1}$); $Y_{k,\beta}$ is $Y_{k,\beta}^*$ with $u = u_m$; $f_{k,\beta}$ is given explicitly in terms of F_m. The relation (4) is proved in ch. (chapter) I and yields (a) $\lambda_1 > 0$, (b) $\int_{t_0}^{t} f_{00} dt_1 < K$ for (1). Ch. II gives e.'s of the $Y_{k,\beta} (\beta \geqq 2)$. One has $\varphi \subset N(B, \tau, P)$ if (1) generalized derivatives $D^p\varphi$ with respect to x_1, $\cdots$, x_n exist, (2) $|D^k\varphi| < B$ $(k < -\tau)$, (3) $\int \cdots \int |D^k\varphi|^{\sigma} d\Omega < B^{\sigma_1}$, $\sigma_1 = n/(k+\tau)$; $-\tau < k < p$. In ch. II it is shown that, if the A_{ij}, ρ and F_m belong to certain classes N, then $Y_{k,\beta} < Le^{\sigma t}$ $(\sigma \geqq 0, k \leqq p)$ and that u_m is bounded if $p \geqq [n/2] + 1$. In ch. III the preceding results are applied to study s.'s of (1), (2). *W. J. Trjitzinsky* (Urbana, Ill.).

Petrovsky, I. Dependence of solutions of Cauchy's problem on initial conditions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 38, 151–153 (1943). [MF 8679]

Consider the linear hyperbolic equation

(1) $$\frac{\partial^n u}{\partial t^n} = \sum_{k_0 + \cdots + k_p \leqq n} a_{k_0 \cdots k_p} \frac{\partial^{k_0 + \cdots + k_p} u}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_p^{k_p}}, \qquad k_0 < n.$$

The lateral surface of its characteristic cone with vertex at point $P = (t^*, x_1^*, \cdots, x_p^*)$ decomposes the plane $t = t_0 < t^*$ carrying the initial data into partial domains. One such partial domain G_{i_0} is called lacuna, if varying the initial data within G_{i_0} leaves the solution of Cauchy's problem invarient at P. The author states, without proof, the necessary and sufficient conditions that equation (1) possesses lacunae. *F. G. Dressel* (Durham, N. C.).

GEOMETRY

Baier, Othmar. Elementarer Beweis der Dreiecksungleichung in der Poincaréschen Halbebene. Math. Z. **48**, 527–529 (1942). [MF 8726]

The proof is based upon the familiar properties of coaxial circles. *G. de B. Robinson* (Ottawa, Ont.).

Labra y Fernández, Manuel. Some general formulae concerning the configuration of Ceva. Revista Soc. Cubana Ci. Fís. Mat. **1**, 73–83 (1943). (Spanish) [MF 8939]

Baker, H. F. A remark on polygons. J. London Math. Soc. **17**, 162–164 (1942). [MF 8264]

Neumann, B. H. A remark on polygons. J. London Math. Soc. **17**, 165–166 (1942). [MF 8265]

Two very simple proofs of the results of J. Douglas [J. Math. Phys. Mass. Inst. Tech. **19**, 93–130 (1940); these Rev. **1**, 261] and B. H. Neumann [J. London Math. Soc. **16**, 230–245 (1941); these Rev. **4**, 51] on the construction of a regular polygon from an arbitrary polygon. *H. Busemann* (Chicago, Ill.).

Rangaswami, K. The geometry of Hart tetrads. J. Indian Math. Soc. (N.S.) **6**, 131–136 (1942). [MF 8296]

If the circles of a plane are mapped by the points of a projective three-dimensional space, so that the point-circles are mapped by points of a quadric surface, then the eight points which correspond to the circles of two complementary Hart tetrads lie on a twisted cubic curve [Vaidyanathaswamy, Proc. Indian Acad. Sci., Sect A. **1**, 67–73 (1934)]. The present paper gives a more detailed specification of this cubic in relation to the representation; for details the reader must be referred to the paper. *J. A. Todd.*

Wu, Ta-Jen. Projectivities on a line and non-Euclidean motions in space. Acad. Sinica Science Record **1**, 59–61 (1942). [MF 8836]

Stephanos [Math. Ann. **22**, 299–331 (1883)] established a 1-1 correspondence between the projectivities on the straight line and the points of projective 3-space by mapping the projectivity given by the matrix

$$x=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

on the point (x_1, x_2, x_3, x_4). Hence the singular projectivities are mapped on the quadratic surface $\Delta \equiv x_1x_4-x_2x_3=0$.

The author studies the non-Euclidean geometry which has Δ as its fundamental surface; some of his results are well known. All of them can be derived at once from elliptic geometry by transforming its fundamental surface into Δ through a complex projectivity [cf. Klein, Nicht-euklidische Geometrie, Springer, Berlin, 1928]. However, by the use of Stephanos' representation, the formulas become particularly elegant. For example, the projectivities that leave Δ fixed have the form $x^*=axb$, where a and b are two-rowed square matrices. However, it should be remarked that this representation is equivalent to the quaternion notation of elliptic geometry [Klein, l.c. pp. 238–240]. In the last part the author states some theorems on conjugate projectivities on the straight line, that is, projectivities that correspond to points which are conjugate with respect to Δ. The details and proofs are promised in a subsequent paper. *P. Scherk.*

Haenzel, G. Geometrie und Wellenmechanik. III. Die Elemente auf der Eigenwertfläche und der Kernfläche. Jber. Deutsch. Math. Verein. **52**, 103–117 (1942). [MF 8539]

[The first two installments appeared in the same Jber. **49**, 215–242 (1940) and **50**, 121–129 (1940); these Rev. **1**, 262; **2**, 152]. Parts I and II of this paper are devoted to the representation of Eddington's operators E_a first by means of a Kummer surface and then via the Pascal and Brianchon configurations in the plane. In this third part, the author constructs an $n_1=\nu$ sheeted surface, on each sheet of which are drawn ν concentric circles of radii $1, 2, 3, \cdots, \nu$ designated by $n_2=0, 1, 2, \cdots, \nu-1$. Representing an initial direction through the common centre by $n_3=0$, he marks on these circles in succession the points corresponding to rotations of $2\pi/2n_3+1$, where $n_3=0, \pm1, \cdots, \pm(\nu-1)$. The sheets $n_1=r$, $r+1$ are connected along the circumference of the circle $n_2=r-1$. On this model the author is able to represent graphically all possible changes of energy level in the atom, a convention being made to include spin. He explains at some length the significance of the model with regard to the arrangement of the elements in the periodic table. *G. de B. Robinson* (Ottawa, Ont.).

Beck, H. Eine Klasse volumentreuer Transformationen. J. Reine Angew. Math. **184**, 1–11 (1942). [MF 9035]

Among all volume preserving mappings of a Euclidean R_n upon itself, the author considers the very special class of mappings $x\rightarrow x^*$, in which the medial point $m=(x+x^*)/2$ describes a given locus. If this medial locus is a fixed point, then obviously the mapping is either a reflexion or a semi-rotation. The author discusses the geometrical properties of mappings whose medial locus is a curve and, in particular, a straight line. He is unable to extend his method to medial loci of dimension higher than the first. *M. S. Knebelman.*

Convex Domains, Integral Geometry

Valeiras, Antonio. The triangle of minimal perimeter inscribed in another triangle and non-Euclidean geometries. Publ. Circulo Mat. Inst. Nac. Profesorado Secund. no. 6, 14 pp. (1942). (Spanish) [MF 8728]

Of all triangles inscribed in a given acute-angled triangle, the one having its vertices at the feet of the altitudes of the given triangle has the least perimeter. Thus it is very easy to show by elementary geometry that no triangle other than the orthocentric one can minimize the perimeter, and by continuity that the greatest lower bound actually is attained. There is a geometric proof by H. A. Schwarz of the above result, independent of considerations of continuity but involving the Euclidean postulate on parallels. The author now presents an alternative geometric proof, independent of the parallel-postulate. It follows that in geometries like those of Lobatchewski and Riemann, in which the parallel-postulate does not hold, the above result can be established by elementary geometric means. *E. F. Beckenbach* (Austin, Tex.).

Dinghas, Alexander. Isoperimetrische Ungleichungen für konvexe Bereiche mit Ecken. Math. Z. **48**, 428–440 (1942). [MF 8719]

The author considers and solves the problem of determining that convex curve with perimeter L, lying in an infinite strip of prescribed width, with corners of prescribed angles at prescribed places, and giving maximum area. Corresponding to any convex curve K with the desired properties, an auxiliary curve K^* is constructed by a geometrical procedure employing parallel tangents. The area and length of K are expressed in terms of K^*, and an examination of the properties of K^* leads immediately to that K^* corresponding to K of maximum area. The construction of this maximizing K is described, being entirely out of circular arcs and lines. A lower bound for L is found in order that the problem have a solution. The corresponding problem is also considered and solved for the case of a triangle instead of an infinite strip.

J. W. Green (Aberdeen, Md.).

Bol, G. Über Eikörper mit Vieleckschatten. Math. Z. **48**, 227–246 (1942). [MF 8550]

The paper deals with convex bodies that have polygons as projections. W. Süss and H. Kneser found [Jber. Deutsch. Math. Verein. **51** (1941), Problem 298] that a convex body having as projections on any plane a quadrangle is itself a solid quadrangle (that is, convex hull of 4 points). The author proves the following theorems, entirely by elementary means. A convex body is a polyhedron if its projection on any plane in any direction is always a polygon. More exactly, if its projections are polygons of at most N vertices, the solid is a polyhedron, each face of which has at most $2N-6$ vertices. (The convex body here must not degenerate into a plane set of points.) If no projection of the body has more than 5 vertices, then the body itself is either a solid pentagon or hexagon; in the latter case the hexagon is either the convex hull of a parallelogram and a segment or obtainable by cutting a tetrahedron by a plane parallel to one face. *F. John* (Aberdeen, Md.).

Gere, Brewster H. and Zupnik, David. On the construction of curves of constant width. J. Math. Phys. Mass. Inst. Tech. **22**, 31–36 (1943). [MF 8768]

Construction of curves of constant width as involutes of a "star" consisting of an odd number of convex curves of total curvature π [see Bonnesen-Fenchel, Konvexe Körper, Springer, Berlin, 1934, p. 132]. The authors derive, as a necessary and sufficient condition for such a star with successive sides of lengths $\lambda_1, \lambda_2, \cdots, \lambda_n$ to be an evolute of a curve of constant width $2a$, the inequalities

$$2a \geqq (\lambda_1+\lambda_3+\cdots+\lambda_n)-(\lambda_2+\lambda_4+\cdots+\lambda_{n-1}),$$

obtained for all possible choices of initial sides.

F. John (Aberdeen, Md.).

Maak, Wilhelm. Schnittpunktanzahl rektifizierbarer und nichtrektifizierbarer Kurven. Math. Ann. **118**, 299–304 (1942). [MF 8930]

The author gives a proof of the so-called Poincaré formula: if K and K' are two continuous plane curves, K fixed and K' movable in a rigid fashion, and N the number of their intersections (properly defined), N being a function of the three parameters of rigid motion x, y, θ, then

$$I=\int\int\int N(x, y, \theta)\,dx\,dy\,d\theta=4LL',$$

where L and L' are the lengths of K and K', respectively, the value L or $L'=+\infty$ being admitted. The proof proceeds by the construction of polygonal approximations to K, while K' is assumed to be simple and closed. It is shown by this means that no change in I results if K is replaced by a line of length L, and so the result is demonstrated for a curve and a line segment. By reversing the roles of K and K' the general theorem is obtained. As a corollary the following theorem is proved: if two curves have a bounded number of intersections for all relative positions, both are rectifiable.

J. W. Green (Aberdeen, Md.).

Algebraic Geometry

Bronowski, J. The figure of six points in space of four dimensions. Quart. J. Math., Oxford Ser. **14**, 5–15 (1943). [MF 8626]

A discussion of cubic primals R in [4] with nodes at six general points, and of the rational quartic curves Q on these six points. The primals are studied by projecting into [3] from a node. Attention is concentrated on the systems of lines in R and the chords and trisecant planes of Q. There is also obtained a representation of the system (r) of cubics R by the prime sections of the Segre cubic K in [4] with ten nodes. Various properties of K and of (r) are obtained from this; in particular, consideration of the images of the reducible quartics Q gives the Segre configuration of nodes and planes on K. *R. J. Walker* (Aberdeen, Md.).

Chariar, V. R. and Chatterji, N. On a certain quartic scroll associated with a pair of given lines and two given quadrics. Bull. Calcutta Math. Soc. **34**, 183–185 (1942). [MF 9000]

A quartic scroll with two skew double lines is defined by a (2,2) correspondence between planes through the lines. The present note considers the case in which the lines meet two quadric surfaces in harmonic points. Within the system of such scrolls are included certain quadrics through the skew quadrilaterals formed by the nodal lines taken twice. The same procedure is then applied to the sextic surfaces of lines meeting three quadrics in pairs of points in involution. These surfaces have three-fold directrices, and are expressed by means of a (3,3) correspondence. *V. Snyder.*

Edge, W. L. The contact net of quadrics. Proc. London Math. Soc. (2) **48**, 112–121 (1943). [MF 9114]

Given a tetrahedron T with vertices at A, B, C, D. A contact net of quadric surfaces is defined as a net which touches a fixed tangent line at each vertex. These lines are not arbitrary; the necessary and sufficient condition that the system form a net is that each line meets the opposite face of T in A', B', C', D' which with the vertices of T form a pair of Möbius tetrads. If T is taken as the tetrahedron of reference, the vertices of T' are their reciprocals. The relations between T and T' were discussed in an earlier paper [Edge, Proc. London Math. Soc. (2) **41**, 337–360 (1936)]. Various additional properties are derived in the present paper by algebraic methods. The equations of the defining quadrics of the net are obtained. The base points of a Möbius net are given in terms of the parameters of the net

by the rows of a certain matrix and of the reciprocals of these elements.

The transversal lines of AA', BB', etc., are transformed into space cubic curves by the same Cremona transformation; these curves all pass through the vertices of T and T'. The Jacobian of the net, locus of the vertices of quadric cones in the net consist of these two cubic curves. The trisecant lines of this composite Jacobian are discussed. Their locus consists of two quartic ruled surfaces, each having one cubic curve as double curve and passing simply through the other. The properties of the net can also be obtained by starting with these two quartic ruled surfaces.

V. Snyder (Winter Park, Fla.).

van der Waerden, B. L. Die Bedeutung des Bewertungsbegriffs für die algebraische Geometrie. Jber. Deutsch. Math. Verein. **52**, 161–172 (1942). [MF 8542]

The author explains the role played by the theory of valuations in algebraic geometry. A discussion is made of the several possible ways to generalize for surfaces (or varieties of higher dimension) the notion of a point on the Riemann surface of a curve. If the aspect of the valuation theory is generalized, one arrives at the notions of prime divisor (valuation of dimension 1) or of what the author calls "Ort" (valuation of dimension 0). If the aspect of function theory is generalized, one gets the notion of place (representation of the coordinates in the neighborhood of a point by analytic functions of two parameters).

The discussion of the notion of prime divisor leads naturally to the question of reduction of singularities. This problem is decomposed into a local problem and into a problem in the large. The local problem has been solved by Zariski [Ann. of Math. (2) **41**, 852–896 (1940); these Rev. **2**, 124]. A previous solution of the problem is credited to Kneser [Jber. Deutsch. Math. Verein. **45**, 76 (1935)]. This attribution of priority seems unfair. Kneser published only a short note in which he outlined the idea of a proof of the local uniformization theorem. Considering the great importance of the result the fact that Kneser never came back to the question makes it seem probable that he ran into serious difficulties in trying to write down the missing details of his proof.

The last section of the paper is concerned with the notion of "place" which was introduced by Jung. It is pointed out that it is not yet known whether the natural conditions which should be imposed on a system of places in the sense of Jung determine such a system or not. *C. Chevalley.*

Zariski, Oscar. Foundations of a general theory of birational correspondences. Trans. Amer. Math. Soc. **53**, 490–542 (1943). [MF 8422]

This is a systematic presentation of the basic properties of birational transformations between arbitrary irreducible algebraic varieties defined over an arbitrary ground field K. Most of the main theorems are extensions to this general situation of known properties of nonsingular varieties over the complex field. The method employed is purely algebraic and is based on the theory of valuations and related concepts.

Part I develops some preliminary material. The relations involved in passing between homogeneous and nonhomogeneous coordinates are pointed out, particularly those connecting the ideals in the corresponding rings $\mathfrak{o}$ and $\mathfrak{o}^*$ of polynomials in the coordinates of a general point of an irreducible variety V. The center of a valuation v in the field Σ of rational functions on V is defined to be the irreducible variety W determined by the homogeneous prime ideal $\mathfrak{P}$ consisting of those forms whose ratio to a sufficiently general form of the same degree has positive value. Various relations are pointed out involving W, its quotient ring and the valuation ring of v. A proof is given that every irreducible subvariety of V is the center of at least one valuation, more generally, of a variety of valuations of specified structure. In part II the general theory of birational correspondences is developed. If two varieties V and V' have the same field Σ over K they are birationally equivalent, and two irreducible subvarieties W of V and W' of V' correspond if there is a valuation in Σ whose center is W on V and W' on V'. Interest centers mainly on the departure of the correspondence from (1,1)-ness. In investigating this it is convenient to pass from V and V' to their derived locally normal models $\bar{V}$ and $\bar{V}'$. [See Zariski, Amer. J. Math. **61**, 249–294 (1939).] A proof is given that such models exist over general ground fields. The birational correspondence between V and $\bar{V}$ is particularly simple; V is in (1,1) correspondence with a projection of $\bar{V}$. Assuming V and V' to be locally normal, the behavior of an irreducible subvariety W of V is classified by means of its quotient ring $Q(W)$; W is (1) regular, (2) irregular, (3) fundamental if there is a W' on V' corresponding to W and such that (1) $Q(W)=Q(W')$, (2) $Q(W)\supset Q(W')$, (3) $Q(W)\not\supseteq Q(W')$. This leads to the following geometric distinction: (1) W has a unique correspondant W', of the same dimension, and W is the only correspondant of W'; (2) W has a unique correspondant W', of dimension not greater than that of W, but W' has other correspondants than W; (3) W has more than one correspondant (in fact infinitely many). The term "irregular" has been introduced by the author to distinguish clearly between behaviors (2) and (3) which have previously been lumped together as "fundamental."

For the further investigation of fundamental varieties the join V^* of V and V' is introduced. Its usefulness is due to the fact that the transformation from V to V' can be made in two steps, V to V^* and V^* to V', and neither of these two correspondences has any fundamental elements on V^*. The transform $T[W]$ of W is the subvariety of V' such that (A) each irreducible component of $T[W]$ corresponds to W and (B) each irreducible subvariety of V' which corresponds to W lies on $T[W]$. That $T[W]$ exists for each W is shown first for the case in which V' is the join V^*, and then extended to the general case. If there are no fundamental elements on V' then $T[W]$ is of higher dimension than W; if in addition V is locally normal at W then each irreducible component of $T[W]$ is of higher dimension than W. The proof of this last fact is quite long, and involves a considerable investigation of the prime ideals in the polynomial rings $\mathfrak{o}$ and $\mathfrak{o}'$ of V and V'. (V' can be taken to be V^*, so that $\mathfrak{o}'\supset\mathfrak{o}$.) The linear system on V associated with the correspondence is constructed, and is shown to have as its base manifold F the set of fundamental varieties of the correspondence. Isolated fundamental varieties are defined (they include the irreducible components of F, together with certain "embedded" subvarieties of these); their transforms, under certain general conditions, are of dimension one less than that of V'. Finally, a general monoidal transformation is constructed, having as its only isolated fundamental varieties a given irreducible subvariety W of V and subvarieties of W. The particular case in which W is a simple subvariety of V is investigated in detail.

R. J. Walker (Aberdeen, Md.).

Differential Geometry

Kasner, Edward. Dynamical trajectories in a resisting medium. Proc. Nat. Acad. Sci. U. S. A. **29**, 263–268 (1943). [MF 9018]

The author studies the geometry of the motion of a particle moving in the plane under a positional field of force and influenced by a resisting medium, the resistance R acting in the direction of motion and varying as some function of the speed v. The differential equation of the ∞^3 dynamical trajectories is obtained in an intrinsic form which is free of the time t. The geometric behavior of these trajectories is then contrasted with the five geometric properties which were previously obtained by the author for the case of a positional force ($R=0$) [Differential-Geometric Aspects of Dynamics, Amer. Math. Soc. Colloquium Publ., vol. 3, part II, New York, 1913 (reprinted 1934)] and which completely characterize this case. It is proved that the only appropriate resisting media for which property I of this characterization is valid are those for which the resistance R is of the form Av^2+B, where A and B are constants. The only medium preserving properties I and II is that in which $R=Av^2$. If property III is also imposed, $R=0$, so that the force is positional. The analogue of property I for the case in which $R=Av^n$ is also considered. Finally, corresponding results for fields of force in Euclidean 3-space are obtained.

A. Fialkow (New York, N. Y.).

DeCicco, John. Dynamical trajectories of the curvature type. Proc. Nat. Acad. Sci. U. S. A. **29**, 268–270 (1943). [MF 9019]

Let F be a family of ∞^4 curves in Euclidean 3-space, one in each direction at each point. A curvature trajectory of F is a curve which, at each point, has the same osculating plane as, and also c times the curvature of, the member of F to which it is tangent at that point, c remaining constant along the trajectory. The totality of curvature trajectories constitute a family of ∞^5 curves in space. The totality of dynamical trajectories associated with any positional field of force in space is also a family of ∞^5 curves. In this paper the author announces the following theorem. Those families of ∞^5 curves which are simultaneously dynamical and curvature trajectories are the dynamical trajectories of the following distinct types of fields of force: (1) those whose lines of force all lie in a pencil of planes, (2) those whose lines of force are orthogonal to a family of ∞^2 circular helices, all of which possess the same axis and the same period, (3) those of the central or parallel type. It was previously shown by Kasner that the corresponding problem in the plane leads to fields of force of type (3) only.

A. Fialkow (New York, N. Y.).

Kasner, Edward and DeCicco, John. A generalized theory of dynamical trajectories. Trans. Amer. Math. Soc. **54**, 23–38 (1943). [MF 8708]

A generalized field of force is one in which the force depends not only on the position of the point but also on the direction through the point. It is shown that the differential equation of the ∞^3 dynamical trajectories associated with a plane generalized field of force is always of the form $y'''=G(x, y, y')y''+H(x, y, y')y''^2$. Conversely, any system of curves which satisfies a differential equation of this type may always be identified (in many ways) with the trajectories of a generalized field of force. It is shown that the following geometric property is a complete characterization of the trajectories of the above differential equation. For each of the ∞^1 curves of the ∞^3 trajectories which pass through a given lineal element E, construct the osculating parabola at E. Then the locus of the foci is a circle through the point of E. Equivalent geometric characterizations are also derived. The authors generalize the concept of "line of force" to apply to a generalized field. If a particle starts at rest from a point P of the field its path (called a "rest trajectory") is tangent to the direction E of the line of force at P. If the curvature of the rest trajectory is not zero at P, then the ratio of the curvatures of the rest trajectory and of the corresponding line of force is equal to $(1-\lambda)/(3-\lambda)$. Here λ is the instantaneous rate of change of the inclination of the force vector for the lineal element E with respect to the inclination of E. The case in which the rest trajectory has higher than first order contact with its tangent at P is also solved. The paper concludes with a discussion of various systems of curves (velocity systems, systems S_h, pressure curves) previously defined in connection with positional fields of force. *A. Fialkow.*

Kasner, Edward and DeCicco, John. Union-preserving transformations of differential-elements. Proc. Nat. Acad. Sci. U. S. A. **29**, 271–275 (1943). [MF 9020]

This paper concerns itself with transformations of plane differential elements of order n

$$(x, y, dy/dx, d^2y/dx^2, \cdots, d^ny/dx^n)$$

into lineal elements $(x, y, dy/dx)$ and the conditions under which these element transformations convert every union (curve) of nth order elements into a union of lineal elements. It is therefore a generalization of the classical Lie theory of contact transformations. It is shown that the only available union-preserving transformations between differential elements of any orders are the Lie contact group of lineal elements, the authors' new set of union-preserving transformations from elements of order n ($n\geqq 2$) into lineal elements or extensions of these two types. The following results for the case $n=2$ (curvature elements to lineal elements) are typical of the general case. (1) Any transformation, not of the union-preserving type, carries exactly ∞^3 unions into unions. If more than ∞^3 unions are converted into unions, the transformation is of the union-preserving type. (2) Every union-preserving transformation (except certain degenerate and special categories) defines a unique directrix equation, and conversely. (3) Any union-preserving transformation may be defined by considering the osculating curves, chosen from among a given parametrized family of ∞^3 curves, to an arbitrary curve. This last result is used to generalize the classical theory of evolutes and involutes of a curve. *A. Fialkow.*

Ladue, Mary Elizabeth. Conformal geometry of horn angles of higher order. Amer. J. Math. **65**, 455–476 (1943). [MF 8704]

The configuration formed by an ordered pair of plane analytic curves which have nth order contact at a common point is called a horn angle of nth order. It had been shown by Kasner that an nth order horn angle has one and only one nontrivial differential invariant (of finite order) with respect to conformal transformations of the plane. For horn angles of orders 1 and 2, the explicit form of these invariants was found by Kasner. In the present paper the corresponding invariants for horn angles of orders 3, 4 and 5 are obtained. The method follows that previously employed, depending upon the use of power series in the neighborhood

of the vertex of the horn angle. However, the detailed calculations of the present paper are of considerably greater complexity than those which occur for simpler (horn) angles. All analytic curves having nth order contact with each other at a common point may be thought of as forming a space in which each of the curves is a "point" and the "distance" between two points is the conformal invariant for the corresponding horn angle. The geometry of this space when $n=3$ is studied. Finally various analogous results are obtained in the equilong geometry of second and third order horn angles which are roughly dual to the corresponding conformal cases. *A. Fialkow* (New York, N. Y.).

Haupt, Otto. Raumbogen mit Punkten von beliebig vorgegebenem linearen Ordnungswert. J. Reine Angew. Math. **184**, 77–90 (1942). [MF 8990]

The cyclic order (c.o.) of a plane curve is the maximum number of its points of intersection with a circle. The (one-sided) c.o. of a point on a curve is the c.o. of a sufficiently small (one-sided) neighborhood of that point on the curve. It is not less than 3. The points of one-sided c.o. 3 have a c.o. not greater than 6, and the points of c.o. 4 have the one-sided c.o. 3. Let $K\geqq 5$ be an integer. The author constructs a plane curve with a tangent everywhere, one of whose end-points has the c.o. K. Projecting this curve on a sphere, he obtains a skew curve with a tangent everywhere, one of whose end-points has the (correspondingly defined) linear order K. *P. Scherk* (Saskatoon, Sask.).

Terracini, Alejandro. Tangents on the curve of contact of a given surface and a ruled surface with straight directrices. Revista Union Mat. Argentina **9**, 1–24 (1943). (Spanish) [MF 8883]

Given a surface and a circumscribing conoid. It is required to construct a tangent line l to the curve of contact of the surface and the conoid at one of its points P. Let P be the line of the congruence through P and r, s the rectilinear directrices of the congruence to which the conoid belongs. The present paper generalizes the problem by considering only the element of the surface. The line p', conjugate of p in the indicatrix at P, is introduced. The construction is first made by the methods of orthographic descriptive geometry with figures; the case is featured in which one of the directrices is the infinite line of the tangent plane, or another well-defined line. Extensive use is made of the von Staudt equation involving segments of the line p to r and s and the radius of curvature of the curve of contact at P. The problem is then interpreted in terms of the Klein map on a quadric primal in [5] and also by means of the Plücker p_{ik} coordinates. As P describes the curve of contact, r, s remain fixed; a net of linear complexes appears, the third defining one being the special complex having p as axis. Finally, a projective invariant is considered, on which a second system, defined by p', r', s', projectively related to the first, is considered. *V. Snyder* (Winter Park, Fla.).

Biran, Lutfi. Les surfaces réglées étudiées en analogie avec les courbes gauches. Rev. Fac. Sci. Univ. Istanbul. Ser. A. **6**, 121–134, erratum, 244 (1941). (French. Turkish summary) [MF 8451]

A ruled surface can be given by giving a "dual vector" (which corresponds to the generator of the surface) as function of a parameter; dual vectors are vectors $a+\epsilon b$ with a, b ordinary vectors and $\epsilon^2=0$. There exist the formulae of Blaschke which have the formal structure of the Frenet formulae for curves, all quantities involved being dual vectors and numbers. It is now proved that the dual curvature and torsion appearing in these formulae determine the surface up to congruence. Then the analogues of the cases usually studied for curves are treated: surfaces with constant dual curvature, surfaces for which there is a linear relation between dual curvature and torsion, surfaces with constant ratio of dual curvature and torsion, etc. *H. Samelson* (Syracuse, N. Y.).

Grüss, Gerhard. Zur anschaulichen Geometrie der Tangentenfläche einer Raumkurve. J. Reine Angew. Math. **184**, 65–76 (1942). [MF 9004]

This paper gives an analysis of the behavior of the tangent surface of a regular space curve C in the neighborhood N of a point p at which the curvature κ vanishes. It is assumed that p is the only point in N at which $\kappa=0$ and that the torsion of C has a positive lower bound in N. Whereas, for a curve with nonvanishing curvature and torsion, the tangent surface consists of two sheets with the curve as common edge (of regression), it is found in the case under consideration that on either side of p the tangent surface consists of three sheets, of which one has C as edge, another has the tangent line T of C at p as edge and the third is bounded by T and C. The method employed is the classical one of determining the intersection of the tangent surface with the normal plane at an arbitrary point of C, in the process of which extensive use is made of power series expansions. *G. A. Hedlund* (Charlottesville, Va.).

Choudhury, A. C. On the Steiner's polygons in a web. J. Indian Math. Soc. (N.S.) **6**, 122–126 (1942). [MF 8294]

The geometry of a three-web as developed by G. Thomsen associates with a web a quasi group and vice versa. This allows the definition of a law of multiplication, in which the associative law does not hold. It is shown that a Steiner polygon in a plane cubic curve can be considered as the geometrical expression of the equation $a^n=1$. An example is obtained in the web $r=m$, $\theta=2n\pi/p$, $r+p\theta/2\pi=m+n$, $1\leqq m\leqq p$, $1\leqq n\leqq p$, m and n integers. Several properties of Steiner polygons can be derived from this representation. *D. J. Struik* (Cambridge, Mass.).

Choudhury, A. C. On a generalisation of Thomsen's triangle in a web. Bull. Calcutta Math. Soc. **34**, 93–98 (1942). [MF 7534]

It is shown that a web gives a geometrical figure of a quasi group, and some closed figures different from Thomsen's triangle are derived. These figures lead to associative commutative laws. This leads, in particular, to some results obtained by D. C. Murdoch [Amer. J. Math. **61**, 509–522 (1939); Trans. Amer. Math. Soc. **49**, 392–409 (1941); these Rev. **2**, 218]. *D. J. Struik* (Cambridge, Mass.).

Strubecker, Karl. Zum Cauchyschen Problem der Differentialgleichung $rt-s^2=K$. Deutsche Math. **6**, 507–524 (1942). [MF 8599]

The author gives a geometric characterization of the integral surfaces of the Monge-Ampère partial differential equation $rt-s^2=K$, where K is a nonzero constant. The method used here depends upon the development of the geometry of the isotropic space (x, y, z) whose distance is given by the singular quadratic form $ds^2=dx^2+dy^2$. The fundamental group, the metric properties and the elements of curve and surface theory of this space are derived.

Among other results, the "principal curvatures" of any surface $z=z(x, y)$ are defined and it is shown that their product (called the "relative curvature" of the surface) is given by the expression $rt-s^2$. Hence the integral surfaces of $rt-s^2=K$ are the surfaces S_2 of constant relative curvature in an isotropic space. Additional properties of S_2, particularly of its asymptotic lines, are derived. By means of these properties and known existence theorems, a geometric solution of the above differential equation is obtained. Some specific examples of this solution are given.

A. Fialkow (New York, N. Y.).

Blaschke, Wilhelm. Über die Differenzialgeometrie von Gauss. Jber. Deutsch. Math. Verein. **52**, 61–71 (1942). [MF 8534]

It is assumed, with Cartan, that products of differentials alternate, hence $dxdy=-dydx$, and that under "exterior differentiation" differentials behave like constants, so that the exterior derivative of a Pfaffian form $\omega=Pdx+Qdy$ is $d\omega=(Q_x-P_y)dxdy$. These prescriptions lead to an elegant derivation of the integral formulas of Gauss and Stokes, the "theorema egregium," the theorem of Gauss-Bonnet and the parallelism of Levi-Civita on a surface. There are also applications to Riemannian spaces, leading to an expression for the curvature tensor. *D. J. Struik.*

Lotze, Alfred. Ein einfacher Weg zu den Differentialinvarianten der Flächentheorie. Jber. Deutsch. Math. Verein. **52**, 49–58 (1942). [MF 8532]

The author's problem is to determine a simple method for obtaining the differential invariants of a regular surface imbedded in a three dimensional Euclidean space. By use of a theorem of Grassmann, the author is enabled to construct various differential invariants (Beltrami's differential invariants, invariants associated with the second fundamental tensor, etc.). In each case, the corresponding tensor formulas for the invariants are stated. It is shown that the method may be extended to higher dimensions. The notation is that of Grassmann. *N. Coburn* (Austin, Tex.).

Bernstein, S. Renforcement de mon théorème sur les surfaces à courbure négative. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 285–290 (1942). (Russian. French summary) [MF 8501]

It is shown that, if $z=f(x, y)$ has continuous second order partial derivatives for all finite x, y and represents a surface with Gaussian curvature K satisfying $K\leqq 0$, $K\not\equiv 0$, then there exists a constant $\delta>0$ such that for any $M>0$ there is an (x, y) at which $|z|-\delta(x^2+y^2)^{\frac{1}{2}}>M$. It follows that, if z satisfies an equation of elliptic type

$$Ar+2Bs+Ct=0, \qquad AC-B^2>0,$$

where A, B, C are arbitrary functions of x, y, z, p, q, r, s, t, and if

$$|z|<(x^2+y^2+h^2)^{\alpha/2},$$

where $\alpha<1$ and h are constants, then z reduces to a constant. The proofs are analogous to those given previously by the author [Communications Math. Soc. Charkow **15**, 38–45 (1915)] for less sharp results with $\delta=0$ and $\alpha=0$. An application was made there to minimal surfaces. The reviewer calls attention to the translation of this earlier paper [Math. Z. **26**, 551–558 (1927)], and also to an elementary proof of the result concerning minimal surfaces by T. Radó [Math. Z. **26**, 559–565 (1927)].

E. F. Beckenbach (Austin, Tex.).

Bell, P. O. New systems of hypergeodesics defined on a surface. Bull. Amer. Math. Soc. **49**, 575–580 (1943). [MF 8888]

As a point P moves along a curve C on a nonruled surface S, the tangents to the curves of one of the families of asymptotic curves on S generate a ruled surface S_1, while the tangents to the other asymptotic curves generate a surface S_2. Let S_ρ and S_σ be arbitrary transversal surfaces of the congruences of the first and second family of asymptotic tangents, respectively. If S_ρ intersects S_1 in an asymptotic curve C of S_1, C is called a ρ-tangeodesic; similarly a σ-tangeodesic is defined. A tangeodesic is a hypergeodesic. By means of these types of hypergeodesics, new geometric characterizations are found for the edges of Green, the directrices of Wilczynski and the projective normal of Fubini. *V. G. Grove* (East Lansing, Mich.).

Su, Buchin. Osculating conics of the plane sections through a point of a surface. Amer. J. Math. **65**, 439–449 (1943). [MF 8702]

This paper carries forward the author's investigations of plane sections through a point O on a surface S of projective space in a direction suggested by recent papers by L. Green [Amer. J. Math. **60**, 649–666 (1938)] and Chenkuo Pa [Univ. Nac. Tucumán. Revista A. **2**, 67–77 (1941); these Rev. **3**, 308]. The development is based on the familiar canonical series expansion for S near O. The fundamental result is the following. Given two nonasymptotic tangents t_1, t_2 to S at O and a plane π_1 through t_1, a quadric Q is completely determined passing through the asymptotic tangents at O and having second order contact there. The osculating conics of the sections of S by π_1 and by a certain corresponding π_2 through t_2 both lie on Q; Q contains a third osculating conic (thus determining a residual tangent t_3) and forms a pencil when π_1 turns about t_1. It is found further that all quadrics of this pencil meet S in a family of curves having t_1, t_2 as two triple-point tangents while t_3 is the common tangent of all the osculating conics each one lying on one quadric of the pencil. Several covariant configurations are derived from t_1, t_2, t_3, Q and special cases considered. A new characterization of the quadric of Wilczynski is one of many results. *J. L. Vanderslice* (College Park, Md.).

Su, Buchin. Note on a theorem of B. Segre. Acad. Sinica Science Record **1**, 16–19 (1942). [MF 8824]

It was shown by Segre that there are $n-1$ projective invariants of contact for two curves in an n-dimensional space having the same osculating spaces of all orders at a common point. In this paper the following generalization is proved. Two curves in a projective n-dimensional space which have a point and the first k ($k\leqq n$) osculating spaces defined at this point in common have $k-1$ projective invariants.

A. Fialkow (New York, N. Y.).

Wilkins, J. Ernest, Jr. The first canonical pencil. Duke Math. J. **10**, 173–178 (1943). [MF 8459]

All canonical lines of the first kind through a point P of a nonruled surface S lie in the canonical plane of P and constitute the first canonical pencil with vertex at P. Certain special lines of this pencil are the directrices of Wilczynski, the axes of Čech, the edges of Green and the projective normal. In this paper the author obtains certain geometric definitions for lines of the first canonical pencil. He shows that each such line except the first axis of Čech can be defined (a) in terms of the cusp-axes of two families of

hypergeodesics which are the extremals of two given integral invariants or (b) as the cusp-axis of a cone of class three defined by the osculating planes of the extremals of the integrals defined in (a). He also finds that these lines are intersections of osculating planes of the projective geodesics in the triple of directions D_k* and uses a construction of Hsiung to obtain a canonical line of the first kind from a given canonical line of the first kind. *T. R. Hollcroft.*

Hsiung, Chuan-Chih. On the curvature form and the projective curvatures of a space curve. Acad. Sinica Science Record 1, 72–74 (1942). [MF 8840].

This paper states without proof geometric characterizations of the projective curvatures of Sannia [Ann. Mat. Pura Appl. (4) 1, 1–18 (1923); 3, 1–25 (1925)] and a certain curvature form for a space curve. *V. G. Grove.*

Hsiung, Chuan-Chih. Contributions to the projective differential geometry of a surface. Acad. Sinica Science Record 1, 62–65 (1942). [MF 8837]

The results of this paper were published in Duke Math. J. 10, 217–237 (1943); cf. these Rev. 4, 257.

Wang, Hsien-Chung. On the projective linear element of a non-holonomic surface. Acad. Sinica Science Record 1, 84–86 (1942). [MF 8843]

The results of this paper are contained in the paper reviewed below.

Wang, Hsien-Chung. On a projective invariant of a non-holonomic surface. Ann. of Math. (2) 44, 562–571 (1943). [MF 8880]

Wang, Hsien-Chung. A projective invariant of a non-holonomic surface. Duke Math. J. 10, 565–574 (1943). [MF 8981]

[The two papers are identical.] If $A_i(x)dx^i=0$ $(i=1, 2, 3)$ is the equation of a nonholonomic surface, asymptotic directions at a point may be defined as for ordinary surfaces. The author assumes that these directions are real and distinct and chooses a canonical projective frame of reference associated with each point of the surface. It is then shown that

$$d\sigma^2=((\omega^3\omega_1^3-\omega^3\omega_1^2)(\omega^1\omega_2^3-\omega^2\omega_2^1))/(\omega^3)^2$$

where the ω's are linear differential forms, is a projective invariant. It is interpreted as the cross-ratio of four points of intersection of the four asymptotic lines with the line of intersection of two neighboring tangent planes. The fundamental theorem proved is that, if two nonholonomic surfaces have the same $d\sigma^2$, then they are projectively equivalent either under a collineation or a correlation. It is not apparent what happens when asymptotic directions coincide or why this theorem is not necessarily true in case of holonomic surfaces. *M. S. Knebelman* (Pullman, Wash.).

Su, Buchin. The projective differential geometry of a non-holonomic hypersurface. Duke Math. J. 10, 575–586 (1943). [MF 8982]

This paper extends previous investigations of the author [Ann. Mat. Pura Appl. (4) 19, 289–313 (1940)]. At a generic point O of a nonholonomic hypersurface V_{n+1}^n in projective $(n+1)$-space a certain canonical projective reference system is determined which is a direct generalization of one used in similar investigations by the author and others. Using these coordinates, properties of sections of V_{n+1}^n by planes through a nonasymptotic tangent at O are studied. For example, the locus of the osculating conics of these sections at O is a hyperquadric (a generalization of Moutard's theorem). The balance of the paper studies sections of V_{n+1}^n by three-spaces S through a given plane π containing O and contained in the tangent hyperplane at O. Each section is a nonholonomic V_3^2. Extensive calculation leads to the equation of the Bortolotti quadric to V_3^2 at O and to the theorem that as S turns about π these quadrics lie on a hyperquadric (partial generalization of Čech's theorem). Finally it is found, S varying as before, that the Bortolotti normal to V_3^2 describes an $(n-1)$-space through O. *J. L. Vanderslice* (College Park, Md.).

Coburn, N. Unitary curves in unitary space. Univ. Nac. Tucumán. Revista A. 2, 159–167 (1941). [MF 6756]

In this paper the arc length in unitary space is defined by

$$(ds)(ds^*)=|ds|^2=a_{\lambda\mu^*}d\zeta^\lambda d\zeta^{\mu^*}.$$

If a curve parameter t can be chosen such that $a_{\lambda\mu^*}d_t\zeta^\lambda d_t\zeta^{\mu^*}$ is constant along a unitary curve k_1, $\zeta^k=\zeta^k(t)$ (conj.), then t is called a natural parameter. It is proved that, if k_1 possesses such a natural parameter, then there exist ∞^1 arc length integrals, which are analytic functions of this parameter, all with the same absolute value. This allows the choice of a unique arc length parameter. If and only if k_1 is a Euclidean curve u_1 can an arc length integral exist which is an analytic function of a curve parameter. It is also shown how a unique arc length parameter can be found when the curve parameter is real. The paper ends with the determination of those k_n into which certain types of u_1 can be imbedded. *D. J. Struik* (Cambridge, Mass.).

Lee, Hwa-Chung. A kind of even-dimensional differential geometry and its application to exterior calculus. Amer. J. Math. 65, 433–438 (1943). [MF 8701]

Instead of a symmetric tensor which usually serves as the fundamental tensor of differential geometry, the author uses a skew symmetric tensor. Whereas in the former case the simplest differential operation is of the second order (it furnishes the curvature tensor), for a skew symmetric tensor of rank two there exists a well-known operation of the first order which leads in an invariant way to another tensor (of rank three). This fact is exploited to construct what may be characterized as an imitation of geometry (for even-dimensional spaces), and in view of the lower order of the operation it is much simpler than ordinary geometry. The author considers "flat spaces" (the "curvature tensor," as the author calls the above-mentioned tensor of rank three, vanishes for them), "conformal spaces" (two spaces are conformal if their fundamental tensors differ by a numerical factor) and "conformal curvature tensors" (the vanishing of one of them is a condition of conformality). An alternative formulation of these conceptions and results is given in terms of Goursat-Cartan exterior forms and exterior derivatives; they are applied to finding a necessary and sufficient condition for the existence of an integrating factor of an exterior differential form of even degree (the form in case $n>4$ must be a factor of its exterior derivative). Finally, in a way analogous to that used by Killing in Riemannian geometry, a necessary and sufficient condition is found for a vector to be the vector of an infinitesimal automorphism of the space under consideration. *G. Y. Rainich.*

de Mira Fernandes, A. Pseudo-extensori. Portugaliae Math. **4**, 41–51 (1943). [MF 8516]

The author presents first the characteristic properties of extensors of one or more than one parameter, referring mainly to A. Kawaguchi [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 1–152 (1940); these Rev. **2**, 22]. Then pseudo-extensors are defined both of one and of more than one parameter. If, under the point transformation $x^i \to \bar{x}^\lambda$, $(V^{\alpha i})$ is a contravariant extensor of degree G, then $[\binom{q}{\alpha} V^{\alpha-\alpha_1 i}]$ is a contravariant pseudo-extensor. If (W_i^α) is a covariant pseudo-extensor, then $[\binom{q}{\alpha} W_i^{\alpha-\alpha}]$ is a covariant extensor. If (V_α^i) is a contravariant pseudo-extensor, then $[(1/\binom{q}{\alpha}) V_{\alpha-\alpha}^i]$ is a contravariant extensor. If $(W_{\alpha i})$ is a covariant extensor, then $[(1/\binom{q}{\alpha}) W_{\alpha-\alpha_1 i}]$ is a covariant pseudo-extensor. There $i=1, 2, \cdots, N$; $\alpha=0, 1, \cdots$; G is the degree of differentiation. These pseudo-extensors differ from the pseudo-extensors of T. Suguri [Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **1**, 143–166 (1941); these Rev. **3**, 19]. *D. J. Struik.*

Hostetter, I. M. Polyadic products. J. Math. Phys. Mass. Inst. Tech. **22**, 93–114 (1943). [MF 8948]

Some properties of tensor algebra are derived in the notation of the polyadic calculus. Formulas are derived for linear dependence of polyadics, for the "norm" of a polyadic, for product formations corresponding to inner and alternate products of tensors and for alternate products of tensors with themselves. The relation to the theory of m-way determinants is indicated. The paper ends with necessary and sufficient conditions that a given polyadic has a simple or t-way multipartite rank r. Reference to older works on polyadics [see, e.g., reviewer's "Grundzüge der mehrdimensionalen Differentialgeometrie, Berlin, 1922] and to corresponding investigations written in the notation of the tensor calculus would have facilitated the study of this paper. *D. J. Struik* (Cambridge, Mass.).

MECHANICS

Brelot, M. Sur les principes mathématiques de la mécanique classique. Ann. Univ. Grenoble **19**, 24 pp. (1943). [MF 8675]

A sketch of the foundations and results of Newtonian rational mechanics "modernized" by employing the notation for Lebesgue-Stieltjes integrals of scalar and vector functions. *P. Franklin* (Cambridge, Mass.).

Haviland, E. K. On the singularities and the qualitative behavior of the solutions of the problem of central forces. Amer. J. Math. **65**, 477–491 (1943). [MF 8705]

In the analysis of the motion of two bodies under the influence of their mutual attractions, if the origin of coordinates is chosen so that it coincides with one of the particles, the problem becomes that of investigating the motion of a single particle under a static central force. Let r denote the distance from the origin. This paper makes an exhaustive analysis of the motion and of the behavior at collision if the force function is of the form $r^{-\lambda}$, where $0<\lambda<+\infty$, $\log r$ or $-r^{-\lambda}$, where $-\infty<\lambda<0$. The analysis of the character of the motions is straightforward, but in the determination of the analytic character of the collisions the author finds it necessary in certain cases to make use of Bohr's theory of almost periodic functions of a complex variable.

G. A. Hedlund (Charlottesville, Va.).

Četajev, N. G. Eine Modifizierung des Gauss'schen Prinzips. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 11–12 (1941). (Russian. German summary) [MF 7699]

Consider a dynamical system consisting of n particles subjected to workless constraints and to applied forces depending upon the positions and the velocities of the particles. Let d be a virtual infinitesimal displacement of the kind admitted in Gauss' principle of least constraint, T_d the work done by the applied forces in this displacement and T_d' the work done by the effective forces. Using Gauss' principle the author shows that the difference T_d-T_d' attains a relative maximum for the actual displacement.

L. Bers (Providence, R. I.).

Kimball, W. S. The horizontal path along the earth's surface of a projectile or plane under constant tangential acceleration. J. Franklin Inst. **236**, 67–79 (1943). [MF 8742]

A second explanation, no more satisfactory than the first [same J. **234**, 453–472 (1942); these Rev. **4**, 116], of the author's theory of an additional term in Coriolis' force due to tangential acceleration. A special study is made of the path of a body all of whose energy of motion is due to constant tangential acceleration. *D. C. Lewis.*

Schneller, E. Anschauliches zum drehsymmetrischen Kreisel. Ing.-Arch. **13**, 113–118 (1942). [MF 8584]

An expository paper in which the usual equations of motion of the gyroscope are derived in terms of the Euler angles. Individual terms are interpreted appropriately as components of moments of centrifugal force, of Coriolis' force and of gyroscopic force. *D. C. Lewis.*

Nicolai, E. L. Influence of friction on motion of gyroscope with Cardan suspension. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 66–71 (1943). [MF 8689]

This note deals with the precessional motion of a gyroscope rotating at high speed in a Cardan suspension under the action of constant external moments M_1 and M_2 (about the axes of rotation of the inner and outer rings of the suspension, respectively). Allowance is also made for the moments of friction L_1 and L_2 on the axes of the suspension rings. It is assumed that the angle of rotation of the inner ring remains near its initial value during the time interval under consideration. On the basis of this assumption the differential equations of motion are written in an approximate form which admits of solution by elementary methods. Various types of motion, resulting from different initial conditions, are discussed. *L. A. MacColl.*

Magnus, K. Das Verhalten der Pendel-Neigungsmesser im bewegten Flugzeug. Luftfahrtforschung **19**, 381–393 (1943). [MF 8928]

The author discusses the mechanics of an instrument whose main component is a pendulum and which is used as an indicator of the roll and pitch of an airplane under various flying conditions. *A. E. Heins.*

Sályi, St. Ein graphisches Verfahren zur Bestimmung der Eigenfrequenzen von Systemen mit mehreren Massen. Ing.-Arch. **13**, 104–109 (1942). [MF 8582]

A graphical method is explained for determining the normal frequencies of a system of masses connected by springs. *H. W. March* (Madison, Wis.).

Heinrich, G. Über die Kompensation der Reibung durch zusätzliche Schwingbewegungen. Z. Angew. Math. Mech. **22**, 136–142 (1942). [MF 8895]

A plate P is subjected to a simple harmonic rectilinear oscillation with a velocity $u=u_0\cos\omega t$. A mass M moves on the plate in the direction of the oscillation. Let v be the absolute velocity of M. It is shown that, if $v\ll u_0$ and is constant, the average value of the Coulomb friction force acting between M and P is proportional to v, provided such average value is taken over a time interval which is large with respect to $(2\pi/\omega)$. The result is extended to the case in which v and u form an arbitrary angle. The calculations involve elliptic integrals in this case. *I. Opatowski.*

Schiel, Friedrich. Der schwimmende Balken. Z. Angew. Math. Mech. **22**, 255–262 (1942). [MF 8909]

Astronomy

Schaub, Werner. Grundlagen und Beispiel für die Ableitung der scheinbaren Bahn eines Doppelsternes aus den relativen rechtwinkligen Koordinaten. Astr. Nachr. **272**, 185–190 (1942). [MF 8573]

The given data are the observed relative coordinates in a binary system. Smooth curves are drawn for Δx and Δy as functions of the time; these curves are adjusted to satisfy the law of areas. The coordinates of the center and the times of periastron and apastron, as well as their positions in the apparent ellipse, are then obtained graphically. Then follows the construction of the apparent ellipse, compatible with the adjusted curves for Δx and Δy. From the apparent ellipse the remaining elements are derived by the standard procedure. The exposition of the method is preceded by an analytical discussion of the problem and followed by an illustrative example. The method requires that the observations cover at least one-half of the period of revolution, which should include the times of periastron and apastron. The author states that the adjustment of the curves is difficult, but the ellipse so obtained will be nearly the best obtainable from the observed data. *D. Brouwer.*

García, Godofredo. On the regularization of the two dimensional three-body problem. Revista Ci., Lima **44**, 471–483 (1942). (Spanish) [MF 8749]

Identical with the paper in Actas Acad. Ci. Lima **5**, 101–113 (1942); these Rev. **4**, 174.

García, Godofredo. Generalization of the formula of Lagrange. Generalization of Sundman's inequality, deduced from the homogeneity of the potential and Birkhoff's auxiliary function for the problems of n bodies and of infinitely many bodies (gravitational or nebulous gas). Revista Ci., Lima **45**, 3–15 (1943). (Spanish) [MF 8752]

Identical with the paper published in Actas Acad. Ci. Lima **5**, 131–144 (1942); these Rev. **4**, 174.

García, Godofredo. On the regularization of the two dimensional three-body problem. I. Revista Ci., Lima **45**, 53–90 (1943). (Spanish) [MF 8755]

García, Godofredo. On the regularization of the two dimensional three-body problem. II. Expansion into a series of the regularizing coordinates and of the time as a function of the auxiliary variable of K. Sundman and T. Levi-Civita. Revista Ci., Lima **45**, 91–133 (1943). (Spanish) [MF 8756]

In the first part the author investigates a regularizing transformation of variables for the planar three-body problem due to A. Rosenblatt [Bull. Sci. Math. (2) **52**, 195–202 (1928)]. The known integrals are expressed in the new coordinates and the differential equations are reduced to canonical form by replacing the time t by the variable u used by Levi-Civita and Sundman [see G. D. Birkhoff, Dynamical Systems, Amer. Math. Soc. Colloquium Publ., vol. 9, New York, 1927, p. 275]. In the second part these canonical equations are integrated by Taylor's series expanded about a point corresponding to a double collision. The computation is carried out up to terms of fourth degree in u, and it is found that certain of the lower degree terms vanish. *W. Kaplan* (Ann Arbor, Mich.).

García, Godofredo. On the regularization of the two dimensional three-body problem. III. Conditions for collision in the three-body problem. Revista Ci., Lima **45**, 9 pp. (1943). (Spanish) [MF 9087]

García, Godofredo. On the regularization of the two dimensional three-body problem. IV. Analytic continuation beyond the collision of two of the bodies. Orbits in the neighborhood of a collision of two of the bodies. Revista Ci., Lima **45**, 13 pp. (1943). (Spanish) [MF 9099]

Continuing papers reviewed above, in part III the author obtains, by use of the energy and area integrals, a relation which must be satisfied by the regularizing variables on an orbit corresponding to collision. A similar result was obtained for the restricted problem of three bodies by Levi-Civita [Ann. Mat. Pura Appl. (3) **9**, 1–32 (1904)]. In part IV the regularized orbit is referred back to the original coordinates and it is shown that the colliding bodies move on curves which have cusps at the collision point.
W. Kaplan (Ann Arbor, Mich.).

Kobold, H. Eine Bemerkung zur Störungstheorie. Astr. Nachr. **272**, 212–213 (1942). [MF 8570]

The writer suggests using a variable T defined by $dT=(1+\varepsilon\cos v')^2dt$, where ε and v' are eccentricity and true anomaly of a perturbing planet. A method is then sketched for setting up a table of perturbations corresponding to values of v (for a disturbed planetoid) and v'.
W. Kaplan (Ann Arbor, Mich.).

Hertz, Hans G. On the theory of the Trojan asteroids. Astr. J. **50**, 121–125 (1943). [MF 8821]

The paper contains an account of the details of a practical development of a theory of the Trojan asteroids outlined by E. W. Brown and C. A. Shook in their book "Planetary Theory," Cambridge University Press, 1933. The direct Saturn perturbations and the effects produced by the deviations of Jupiter's orbit from an elliptic orbit have not been considered in this paper. The theory by Brown and Shook uses canonical variables, which permit a reduction to a simplified set of equations of motion from

which the short period terms are absent. On the other hand, a literal form of the development of the disturbing function is necessary. The author shows that the convergence is unsatisfactory if the libration is large. On this account a literal development in powers of the libration is not advisable. A numerical development in Fourier series in multiples of the libration argument improves the convergence. This has been shown by considering orbits in which in addition to the inclination either the eccentricity or the libration is zero. The combination of such orbits might be used to construct a more complete solution. The author has made an experiment with such a procedure in polar coordinates. *D. Brouwer* (New Haven, Conn.).

Gliese, W. Abschätzungen des Kraftfeldes der galaktischen Rotation. Astr. Nachr. **272**, 201–207 (1942). [MF 8568]

Following a suggestion due to Oort [Bull. Astr. Inst. Netherlands **3**, 275 (1927)] that the gravitational field in the galaxy can be approximated by the superposition of the fields due to a central mass (M_1) and a homogeneous oblate spheroidal distribution of mass (M_2), the author investigates the nature of the orbits described by a star in an axially central field of the form

$$V=-GM_1/r+\alpha_1(x^2+y^2)+\alpha_2 z^2+\text{constant}.$$

In the foregoing expression V denotes the gravitational potential, G the constant of gravitation, α_1 and α_2 two constants related to the mass, oblateness and radius of the homogeneous distribution of mass. Orbits confined to the galactic plane ($z=0$) and passing through a representative point ("the solar neighborhood") and appropriate for the discussion of the observed motions of certain special classes of stars are numerically integrated and examined in their relevant astronomical contexts. These orbits are also referred to the frame of a circular orbit passing through the point considered and further illustrated.

S. Chandrasekhar (Williams Bay, Wis.).

Chandrasekhar, S. Dynamical friction. III. A more exact theory of the rate of escape of stars from clusters. Astrophys. J. **98**, 54–60 (1943). [MF 8787]

Continuing previous work on dynamical friction [same J. **97**, 255–262, 263–273 (1943); these Rev. **4**, 260], the author solves the diffusion equation without making the approximation of the previous papers that the coefficient of dynamical friction is independent of velocity; the expression for the friction coefficient in terms of the velocity was derived in the first paper. From the solution an expression for the half-life of a star cluster is obtained. In the case of the Pleiades this is of the order of 3×10^9 years, which is in better agreement with experimental evidence than the previous results. *W. Kaplan* (Ann Arbor, Mich.).

Schmeidler, F. Zur statistischen Verwertung von Sternzahlen und mittleren Parallaxen. Astr. Nachr. **273**, 1–20 (1942). [MF 8571]

Two of the principal problems of stellar statistics are the determination of the stellar density distribution and the frequency of occurrence of the different luminosities L among the stars. The former is specified by a function $D(r)$ giving the number of stars per unit volume at a distance r from the sun (and in a specified direction and confined to an element of solid angle $d\omega$) while the latter is expressed in terms of the "luminosity function" $\varphi(M)$ governing the distribution over the different absolute magnitudes M (proportional, apart from an additive constant, to $2.5\log_{10}L$). For the determination of these functions we have at our disposal the observed number of stars $A(m)dm$ in the interval of apparent magnitude $(m, m+dm)$ and the so-called mean parallaxes $\bar{\pi}(m)$. The quantities $A(m)$ and $\bar{\pi}(m)$ are related to $D(r)$ and $\varphi(M)$ through the integral equations (in the absence of space absorption and certain other simplifying assumptions)

$$(1)\qquad A(m)=\int_0^\infty r^2D(r)\varphi(m+5-5\log_{10}r)dr$$

and

$$(2)\qquad p(m)=A(m)\bar{\pi}(m)=\int_0^\infty rD(r)\varphi(m+5-5\log_{10}r)dr.$$

By an appropriate choice of variables both equations can be reduced to the form

$$(3)\qquad A(m)=\int_{-\infty}^{+\infty}f(\rho)\varphi(m-\rho)d\rho.$$

The problem now is to determine $f(\rho)$ given $A(m)$ and $\varphi(m)$. It has been customary to approximate (3) by a large number of linear equations and solve them numerically. The author treats (3) directly and suggests what appears to be a very useful alternative method of solving it. Basic to this new method is the assumption that (4) $\varphi(x)=0$, $x<\mu$, where μ is some constant (which means that no stars with brightness greater than a certain amount exist). On this assumption we derive from (3) an integral equation of the second kind:

$$(5)\qquad A'(m)=f(m-\mu)\varphi(\mu)+\int_{-\infty}^{m-\mu}f(\rho)\varphi'(m-\rho)d\rho,$$

where primes denote derivatives. The author shows that the solution can be expressed in the form

$$(6)\qquad f(m-\mu)\varphi(\mu)=A'(m)-\int_{-\infty}^{M-\mu}A'(\mu+\eta)\psi(m-\eta)d\eta,$$

where the nucleus $\psi(M)$ is the "reciprocal" of the luminosity function $\varphi(M)$ and is related to it by

$$(7)\qquad \varphi(\mu)\psi(M)+\int_0^{M-\mu}\varphi'(M-\eta)\psi(\mu+\eta)d\eta=\varphi'(M).$$

The advantage of this reduction of (5) is that, once the reciprocal of the luminosity function has been determined, the determination of f for any region of the sky can be effected in terms of just one quadrature. The author determines numerically the reciprocal of the currently adopted luminosity function of van Rhijn [Publ. Kapteyn Astron. Labor. Groningen **38** (1925)] and illustrates the use of equation (6) by considering certain examples.

S. Chandrasekhar (Williams Bay, Wis.).

Stepanov, W. E. Deductions from the vortex theory of sun-spots. Bull. Tashkent Astr. Observatory **2**, 145–157 (1940). (Russian) [MF 8522]

This paper is based on the combined vortex theory by Rankin and works out a mechanical theory of sun-spots. The author finds expressions for the shape, depth and duration of sun-spots. *G. Krotkov* (Kingston, Ont.).

Stepanov, W. E. Zodiacal light. Bull. Tashkent Astr. Observatory **2**, 158–165 (1940). (Russian) [MF 8523]

Solar rays falling on the diffused matter in the interplanetary space cause radiations observed as "zodiacal

light" and "Gegenschein." Observed color-index of the zodiacal light shows that the scattering particles are chiefly large-sized meteoric dust and electrons. Low percentage of polarization as estimated by Dufay in 1925 indicates that the part played by electrons is small. Assuming that the density of the distribution of the particles is a function of the distance from the sun only and if $f_1(r)$ and $f_2(r)$ denote the densities of the electrons and the large particles, respectively, Stepanov gets:

$$f_1(r)=0.453\times10^{-22}(1/\mu_0)[\varphi_1(r)+\varphi_2(r)],$$
$$f_2(r)=(2.7\times10^{-22}/(2/3\pi)A_0)\{\phi_2[2.1(r-0.32)]+0.222\phi_2[19.8(r-0.95)]\}.$$

The second equation shows that the distribution of the large particles has one "definite" maximum in the region of Venus' orbit and one "possible" maximum near the Earth's orbit. Thus there is an inner ring of meteoric dust at the distance of Venus' orbit. The distribution of the small particles is quite different since their density is steadily increasing towards the sun. The probable conclusion is that the electrons are ejected from the sun, while the large meteoric dust has an origin independent of the sun.

G. Krotkov (Kingston, Ont.).

Hydrodynamics, Aerodynamics

Guderley, G. Starke kugelige und zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes bzw. der Zylinderachse. Luftfahrtforschung **19**, 302–311 (1942). [MF 8273]

This investigation starts with the usual assumption of the equation of continuity, the Euler equation and the constancy of entropy and the boundary condition at the (compression) shock wave front for either a spherical or cylindrical wave in a nonviscous fluid. The equations are made amenable to analysis by further assuming (a) $r=c(-t)^n$ for the shock wave front in the neighborhood of $r=t=0$ and (b) $v_1-u\gg a$, where v_1 is fluid velocity on one side, u the shock wave velocity and a the velocity of sound (that is, intense shock waves). The equations reduce to a system of ordinary differential equations and this system is eventually reduced to a single equation in two associated variables by making use of dimensional considerations (which amount to determining certain transformation groups leaving the system invariant). For $n=1$ there is marked formal similarity of the equations to those occurring in the two dimensional stationary flow problem treated by A. Busemann [Luftfahrtforschung **19**, 137–144 (1942); these Rev. **4**, 118]. Plausible physical considerations based on the finiteness of u and a indicate a unique value for n. Thus, for spheres $n=.717$. The solutions with this n value are discussed in some detail and the writer points out explicitly the not surprising lack of accord with the approximations made in the theory of sound for the treatment of weak pulses in a stationary fluid.

D. G. Bourgin (Urbana, Ill.).

Thomas, T. Y. On the uniform convergence of the solutions of the Navier-Stokes equations. Proc. Nat. Acad. Sci. U. S. A. **29**, 243–246 (1943). [MF 8955]

This paper is concerned with the stability problem for the motion of an incompressible viscous fluid. Let $\bar{u}^\alpha$ ($\alpha=1, 2, 3$), $\bar{p}$ denote, respectively, the contravariant velocity components and the pressure of a particular fluid motion in a finite region R; let u^α, p denote the corresponding functions for another fluid motion in R; finally, let $\xi^\alpha=u^\alpha-\bar{u}^\alpha$, $\xi^4=p-\bar{p}$ denote the components of the disturbance. The purpose of the paper is to derive conditions under which the disturbance will converge uniformly to zero in R. By integrating the inequality

$$(1)\qquad g_{\alpha\beta}g^{\sigma\tau}(\xi^\alpha_{,\sigma}+g_{\gamma\sigma}\xi^\alpha f^\gamma)(\xi^\beta_{,\tau}+g_{\delta\tau}\xi^\beta f^\delta)\geqq 0$$

(where $\xi^\alpha_{,\sigma}$ denotes the covariant derivative of ξ^α and f^α denotes an arbitrary vector possessing continuous first derivatives in R) over R and use of the energy equation of the disturbance, the author obtains

$$(2)\qquad dK/dt+\int[\mu(f^\alpha_{,\sigma}-g_{\gamma\delta}f^\gamma f^\delta)g_{\alpha\beta}+\rho D_{\alpha\beta}]\xi^\alpha\xi^\beta dV\leqq 0$$

(where K is the kinetic energy of the disturbance and $D_{\alpha\beta}$ are the components of the deformation tensor of motion $\bar{u}^\alpha$, $\bar{p}$). With the aid of (2), the author shows that "any motion $\bar{u}^\alpha$, $\bar{p}$ is stable in the region R relative to arbitrary finite disturbances provided that its deformation tensor D is sufficiently small." As an example the author considers the stability of Poiseuille motion in a circular pipe relative to arbitrary spatially periodic disturbances. It is shown that if the Reynolds number does not exceed 13 then the motion will be stable.

N. Coburn (Austin, Tex.).

Sircar, H. On H. Poncin's problem. Bull. Calcutta Math. Soc. **34**, 201–213 (1942). [MF 9003]

The functions $f_i(u)$ of Poncin are linearly independent functions defined by the equations

$$f_i(u)x_i^4J_0(x_i)=\mu[J_0(x_i\sqrt{u})-J_0(x_i)],\quad i=1, 2, 3, \cdots,$$

where the quantities x_i are the nonzero roots of the equation

$$w(x)=2\mu J_1(x)-x(x^4+\mu)J_0(x)=0.$$

A simple proof is given of the linear independence of the functions and of the impossibility of initial equilibrium. To study the unsteady motion which sets in when a steady flow breaks down the author derives an expansion

$$1-u=2\mu\sum_{i=1}^{\infty}\frac{x_i^2f_i(u)}{12px_i^2-(x_i^4+\mu)^2}$$

by considering the contour integral

$$(2\pi i)^{-1}\int[J_0(z\sqrt{x})-J_0(z)]dz/w(z).$$

Convergence theorems are given to justify the differentiations by which the basic integrodifferential equation was obtained for the determination of the variable motion.

H. Bateman (Pasadena, Calif.).

Lamla, Ernst. Die symmetrische Potentialströmung eines kompressibeln Gases um einen Kreiszylinder im Freistrahl im unterkritischen Gebiet. Luftfahrtforschung **19**, 358–362 (1943). [MF 8924]

The method of Janzen and Rayleigh is applied to calculate the two-dimensional case of a circular cylinder in a free stream. In the first approximation (incompressible flow) the complex potential is calculated by the method of images, and an expression is obtained that neglects terms of order δ^5, where δ/π is approximately equal to the ratio of cylinder radius to stream breadth. It is shown that this function describes the flow about a nearly-circular cylinder in a free jet. The second approximation, involving terms in α^2, where α is the Mach number of the undisturbed flow, is then calculated. The maximum local velocity at the cylinder is evaluated in terms of δ and α, and the result is compared

with the analogous value for the case of a circular cylinder in a channel [Lamla, Luftfahrtforschung **17**, 329–331 (1940); these Rev. **2**, 266]. The distortion of the stream boundaries is also calculated approximately.

W. R. Sears (Inglewood, Calif.).

Guderley, G. Rückkehrkanten in ebener kompressibler Potentialströmung. Z. Angew. Math. Mech. **22**, 121–126 (1942). [MF 8892]

The author discusses the appearance of edges of regression (R) (envelopes of characteristics) in two dimensional supersonic flows. The arguments are qualitative and supported by idealized diagrams. For compression R's there is a shock wave. The phenomena are quite different for expansion R's; for instance, flow guides are required in addition to the boundary conditions. Since entropy considerations are not mentioned, the physical significance of the expansion R discussion is not clear to the reviewer. *D. G. Bourgin.*

Neronoff, N. Über die wirbelfreie stetige zweidimensionale Bewegung einer unendlichen reibungslosen Flüssigkeit um einen unbeweglichen Zylinder. Z. Angew. Math. Mech. **20**, 329–335 (1940). [MF 8668]

The cylinders considered are those that are transformed into a segment of the real axis and the region exterior to which is mapped on the entire w plane by means of the transformation

$$z=c_0+c_1w+\sum_{p+q=n}[c_n(p,q)w^{p/n}(w-w_0)^{q/n}$$
$$+c_n'(p,q)w^{p/n}(w-w_1)^{q/n}+c_n''(p,q)(w-w_1)^{p/n}(w-w_0)^{q/n}],$$

where w_0, w_1 are real and $0<w_1<w_0$, and also $p\geqq 1$, $q\geqq 1$, $n\geqq 2$. Such cylinders may have re-entrant corners on their contours. The variable $w=\varphi+i\psi$ is interpreted as the complex potential and $\psi=0$ as the streamline that includes the cylinder outline. The author proceeds to determine parametric equations for the arcs forming the cylinder and for the entire streamline $\psi=0$. An example is worked out, and the fluid velocities at the cylinder are calculated for this case. *W. R. Sears* (Inglewood, Calif.).

Görtler, H. und Wieghardt, K. Über eine gewisse Klasse von Strömungen zäher Flüssigkeiten und eine Kennzeichnung der Poiseuille-Strömung. Math. Z. **48**, 247–250 (1942). [MF 8551]

The authors consider a steady plane flow of an incompressible fluid such that the frictional force (kinematic viscosity times the Laplacian of the velocity) possesses a scalar potential. They show that the known solutions comprise all the possible flows satisfying these conditions. In particular the Poiseuille flow is the only plane stationary flow in which the frictional force possesses a nonconstant potential. These results follow from a classification by Hamel of those flows whose stream lines are the contour lines of harmonic functions. In the present problem the Laplacian of the velocity potential is a harmonic function whose contour lines coincide with the stream lines of the flow.

A. H. Copeland (Ann Arbor, Mich.).

Rosenblatt, Alfred. On motions of viscous fluids which are symmetric with respect to an axis. Actas Acad. Ci. Lima **5**, 145–159 (1942). (Spanish) [MF 8374]

The author considers the three dimensional motion of a viscous incompressible fluid, assuming axial symmetry for the motion. The z-axis is taken as the axis of symmetry and the velocity at any point P is resolved into a component p, in the plane through the given point P and the z-axis, and a component H perpendicular to this plane. Upon introducing the stream function ψ and eliminating the pressure and the external force potential from the Navier-Stokes equations, the author obtains a system of two partial differential equations. It is noted that, if

$$\psi=-r^2F(z),\quad H=r\Phi(z), \tag{1}$$

where r is the radial distance from the point P to the z-axis and $F(z)$ and $\Phi(z)$ satisfy

$$\nu F^{iv}+2FF'''+2\Phi\Phi'=0, \tag{2}$$

$$\nu\Phi''-2F'\Phi+2F\Phi'=0, \tag{3}$$

then the above-mentioned system of partial differential equations are satisfied. Boundary conditions for F and Φ are given and an expression for the pressure is obtained. The details of many of the calculations are merely sketched.

The second section of the paper is devoted to a study of equations which are equivalent to (2), (3). A formula is given for determining $F(z)$ when $\Phi(z)$ is known.

N. Coburn (Austin, Tex.).

Rosenblatt, Alfred. On motions of viscous fluids which are symmetric with respect to an axis. Revista Ci., Lima **45**, 17–43 (1943). (Spanish) [MF 8753]

Identical with the paper reviewed above.

Kuo, Y. H. The flow of a compressible viscous fluid through a straight pipe. J. Math. Phys. Mass. Inst. Tech. **22**, 13–30 (1943). [MF 8767]

The author considers the problem of determining the pressure p and the velocity vector u_i for the subsonic flow of a viscous fluid within a straight tube of any constant cross-section. It is assumed that the pressure p and the density ρ are connected by either an adiabatic or an isothermal relation. The author's procedure is as follows. First, p, $1/\rho$ and the momentum vector $V_i=\rho u_i$ are expanded in terms of ascending even powers of the Mach number (supposedly small). Next, the power series for $1/\rho$ and V_i are substituted into dimensionless forms of the equations of continuity and motion, p having been eliminated from the latter equations. Thus the author obtains systems of partial differential equations for the zero, first and second approximations to V_i. However, the boundary conditions of zero velocity at the wall of the tube and known pressure drop are insufficient to solve these partial differential equations. Hence, "normalizing" assumptions are formulated. With the aid of the latter, the author shows that a zero and a first approximation to V_i can be obtained by solving two Dirichlet problems; a second approximation can be obtained by solving a Dirichlet and a biharmonic problem. Formulas for the corresponding approximations to p are obtained. In particular, the zero approximation to p furnishes the well-known Poiseuille equation. Finally, formulas for the drag coefficient and a flux coefficient are obtained. The question of the convergence of the approximations is left open.

In the second section specific application of the preceding results is made to the case of a pipe of circular cross-section. That is, the partial differential equations involved in the zero, first and second approximations are solved. In particular, it is shown, by an examination of the flux coefficient, that, for a given pressure head, the flux is increased by compressibility. *N. Coburn* (Austin, Tex.).

Müller, Wilhelm. Ebene Spannungs- und Strömungsfelder mit zwei kreiszylindrischen Grenzen. Ing.-Arch. **13**, 37–58 (1942). [MF 8598]

The well-known analogy between the stream function of a plane flow of a viscous fluid, in which the inertial forces are neglected, and the Airy stress function of an elastic slab under forces in its own plane is used for the combined treatment of the elastic and hydrodynamic problems of a plane region bounded by two circles (or one circle and one straight line). Bipolar coordinates are used throughout. New proofs are given of many well-known results and some hitherto unsolved special cases are solved and discussed.

P. Neményi (Pullman, Wash.).

Frössel, W. Berichtigungen zu: Berechnung der Reibung und Tragkraft eines endlich breiten Gleitschuhes auf ebener Gleitbahn. Z. Angew. Math. Mech. **22**, 176 (1942). [MF 8905]

This paper appeared in the same Z. **21**, 321–340 (1941); these Rev. **4**, 229.

Chaplygin, J. S. Gliding on the surface of a fluid of limited depth, comparison of the linear and non-linear theories. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 223–252 (1941). (Russian. English summary) [MF 7708]

The author considers the gliding of a plate on a surface of an incompressible fluid, which leads to a problem of flows with a free boundary. He uses Kirchhoff's method. Let w denote the complex potential and let $u(z)$ map the domain in which the flow is defined into a rectangle. The author expresses (dw/du) by a combination of elliptic functions and indicates how to determine the empirical constants. The present investigation of the problem, which has been studied by many authors, was undertaken with a view to checking the solution obtained by the method of thin airfoils and to determining the influence of the depth of the stream on the lift.

S. Bergman (Providence, R. I.).

Rotta, J. Luftkräfte am Tragflügel mit einer seitlichen Scheibe. Ing.-Arch. **13**, 119–131 (1942). [MF 8585]

The author considers a lifting system consisting of a straight horizontal wing of span $b=b_1+b_2$ with inclined shields attached symmetrically above and below it at a distance b_1 from one tip. The inclination of the upper shield to the vertical is $\pi k/2$, and that of the lower shield $-\pi k/2$. The length of each shield is $b_s/2$. It is desired to calculate the lift (vertical force) and induced drag of this system and the centers of pressures of the normal loads on the various panels, all under the conditions of minimum induced drag. The problem has practical importance since it represents the cases of (1) an airplane empennage (rotated 90°) in yawed flight, (2) (with $b_1=0$) a wing with one tip shield, (3) (with $b_1=b_2=0$) a wing with dihedral angle (rotated 90°) in yawed flight.

The distribution of potential at minimum induced drag is determined according to the Prandtl lifting-line wing theory; that is, the problem is reduced to that of two-dimensional flow about a system of barriers having the same form as the lifting system. This is handled by means of a conformal transformation. The integrals giving the force and moment components cannot be evaluated in general; however, detailed results are obtained in certain special cases: (i) $k=0$, (ii) $b_s=0$, (iii) with $k=0$, large values of b_s/b. The last case is intended to represent approximately a straight wing of span b_s attached to a fuselage of height b, in yawed flight.

[It seems to the reviewer that the assumption of minimum induced drag may impose severe restrictions on the general applicability of the results, at least as regards the distribution of the forces.] *W. R. Sears* (Inglewood, Calif.).

Golousin, G. On the theory of the airfoil in the two-dimensional flow. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 146–151 (1943). (Russian. English summary) [MF 8799]

The author compares two plane potential flows around two wings bounded by the curves C_i, $i=1, 2$. It is assumed that for both flows the stagnation points and the magnitude of the velocity at infinity are the same. For the case where C_1 and C_2 are analytic curves sufficiently near to each other the author obtains an approximate formula for P_1-P_2, P_i being the magnitude of the lift for the flow around C_i. This formula (which involves the analytic function mapping the C_1 exterior into that of C_2) is applied to the case where the lower (upper) part of C_2 connecting the two stagnation points lies outside (inside) the wing bounded by C_1. It is shown that

$$(1) \qquad P_1-P_2 \geqq 4\pi\rho V^2|R_1-R_2|.$$

Here ρ is the (constant) density, V the velocity at infinity and R_i the conformal radius of the wing bounded by C_i. (1) holds also for the case where C_i are Jordan curves. That $P_1 \geqq P_2$ has been shown previously by M. A. Lavrentiev [Rec. Mat. [Mat. Sbornik] **41**, 157–165 (1934)].

L. Bers (Providence, R. I.).

Tsien, Hsue-Shen. Symmetrical Joukowsky airfoils in shear flow. Quart. Appl. Math. **1**, 130–148 (1943). [MF 8804]

The velocity components of the fluid far from the airfoil are given by $u=U_0(1+Ky/c)$, $v=0$, where c is the chord of the airfoil, U_0 and K are constants, u and v are velocity components in the directions of the coordinates x and y. The solution is sought in the form of the stream function $\psi=\psi_0+\psi_1$, where

$$\psi_0=U_0(y+(K/2)(y^2/c))$$

and ψ_1 satisfies Laplace's equation. A general expression for ψ_1 for vanishing disturbance velocities at points far from the origin is written, and the flows due to a source, a vortex and a solid circular cylinder in shear flow are considered as examples. Typical streamline patterns are shown for these cases. From the Eulerian equations of motion the author obtains the expression for $\nabla^2 p$ in terms of the parameter U_0K/c and derivatives of ψ_1. The general form of ψ_1 is introduced and the appropriate solution for the pressure p is obtained. By integration around a contour enclosing the body, expressions are obtained, analogous to the Blasius formulae, for the force and couple on any cylinder in this type of flow. These formulae are applied to the case of a symmetrical Joukowsky airfoil. The method of conformal transformation is employed in the determination of ψ_1. The boundary condition of tangential flow at the airfoil surface must be satisfied by the total flow in the airfoil plane, but this condition leads to a boundary condition for ψ_1 in the transformed plane. The Kutta-Joukowsky condition of finite velocity at the trailing edge also leads to a condition on ψ_1 in this plane. From these conditions and the general expression for ψ_1, the circulation and the strengths of the doublets and quadruplets required for the force and

moment are determined. Hence, the formulae for lift and moment coefficient are obtained. These involve, in addition to the usual (potential-flow) terms, terms proportional to K and K^2.

The ten functions that appear in the expressions for the lift and moment coefficients are tabulated for values of the thickness ratio δ between 0 and 1. The aerodynamic-center position and the coefficient of the moment about the aerodynamic center are also calculated and are presented graphically as functions of K and δ. *W. R. Sears.*

Richter, Werner. Das Abwindfeld hinter Tragflügeln mit Klappenausschlag. Luftfahrtforschung **20**, 69–76 (1943). [MF 9041]

The downwash angle at points in a transverse plane located one-half span behind the lifting line has been computed. The trailing-vortex sheet is assumed to be plane, but its location relative to the lifting line is estimated by calculating the downwash angles at a few intermediate points and performing an approximate numerical integration. The computations have been made for three wings of aspect ratio 6, one rectangular and two trapezoidal, each untwisted and also with half-span flaps. *W. R. Sears.*

Helmbold, H. B. und Keune, F. Profilsystematik und Singularitätenverfahren. Beiträge zur Profilforschung. I. Luftfahrtforschung **20**, 77–80 (1943). [MF 9042]

Helmhold, H. B. und Keune, F. Beiträge zur Profilforschung. II. Geometrie der Profilsystematik. Luftfahrtforschung **20**, 81–96 (1943). [MF 9043]

Helmbold, H. B. und Keune, F. Beiträge zur Profilforschung. III. Berechnung der Geschwindigkeitsverteilung in erster Näherung, mit einer Korrektur in der Umgebung der Profilnase. Luftfahrtforschung **20**, 152–170 (1943). (6 plates) [MF 9126]

In the first two papers the authors describe a family of airfoil sections which is modeled on the pattern of the well-known NACA series but comprises a greater variety of forms. The discussion is mostly of a geometric nature; little attention is paid to the aerodynamic characteristics of the profiles.

As is well known, the steady two-dimensional flow of an incompressible fluid past a thin, slightly cambered airfoil profile can be considered as the result of the superposition of the undisturbed uniform flow and the flows induced by certain distributions of vortices, sources and sinks along the mean camber line of the profile. In the third paper formulae are developed which permit the determination of these distributions for the airfoils of the family discussed in the preceding papers. Numerous charts are given which facilitate the computations. *W. Prager* (Providence, R. I.).

Helmbold, H. B. Einige einfache Formeln aus der Tragflügeltheorie. Luftfahrtforschung **19**, 363–366 (1943). [MF 8925]

The purpose of this investigation is to develop a simple approximate relation between the geometrical parameters of the wing and the span-wise lift distribution. If ξ is the distance from the center of span divided by half-span, t the chord, $t_0=t(0)$, α the local geometrical angle of attack, α_p the local effective angle of attack, C_a the local lift coefficient, $C_a(0)=C_{a_0}$, C'_{a_p} the local lift slope, $C'_{a_p}(0)=(C'_{a_p})_0$, b the span, $\kappa=(C'_{a_p})_0/C'_{a_p}$, the integral equation to be solved is

$$(1)\qquad \alpha_p(\xi)+\frac{(C'_{a_p})_0 t_0}{4\pi b}\int_{-1}^{+1}\frac{d}{dx}\left(\alpha_p\frac{\kappa t}{t_0}\right)\frac{dx}{\xi-x}=\alpha(\xi).$$

Following A. Betz [Durand-Betz, Aerodynamic Theory, vol. 4, New York, 1935, p. 56], the author put

$$\alpha_p(\xi)=\frac{C_{a_0}}{(C'_{a_p})_0}\frac{t_0}{\kappa t}(1-\xi^2)^{\frac{1}{2}}(1+k_1\xi^2+k_2\xi^4+\cdots)$$

for symmetrical lift distribution. Then (1) is reduced to an infinite number of algebraic equations for $k_1, k_2, \cdots$. The author's approximate solution is essentially a first order iteration obtained by first assuming $k_2=k_3=\cdots=0$. Calculations are made for untwisted and twisted wings. For the twisted wing, the lift is divided into two parts. One part is proportional to $\bar{\alpha}=\alpha_p(0)$. The other part is due to twist of the wing and gives zero resultant lift for the whole wing. For both cases very simple relations are obtained which are shown to give sufficiently accurate results for preliminary engineering design. *H. S. Tsien* (Pasadena, Calif.).

Schulz, G. Der Abwind auf der Längsachse des Flügels bei Betzscher Zirkulationsverteilung. Luftfahrtforschung **19**, 367–373 (1943). [MF 8926]

Following Helmbold [see the preceding review], the circulation $\Gamma(\xi)$ is expressed as

$$\Gamma(\xi)=\Gamma_0(1-\xi^2)^{\frac{1}{2}}(1+k_1\xi^2+k_2\xi^4+\cdots),$$

where $\Gamma_0=\Gamma(0)$. The induced velocity at a point on the axis of symmetry of wing is calculated. This velocity is divided into two parts: (1) due to the bound vortex, (2) due to the trailing vortices. The trailing vortices are assumed to follow the undisturbed wind velocity and to remain as a vortex sheet without rolling up into a single vortex. The results are expressed in elliptical integrals and values tabulated. This paper differs from that of Helmbold in that actually only k_1 and k_2 are assumed to be different from zero, while in Helmbold's investigation k_3, k_4 are also approximately taken into account. *H. S. Tsien* (Pasadena, Calif.).

Schmidt, Harry und Schröder, Kurt. Die Prandtlsche Grenzschichtgleichung als asymptotische Näherung der Navier-Stokesschen Differentialgleichungen bei unbegrenzt wachsender Reynoldsscher Kennzahl. Deutsche Math. **6**, 307–322 (1942). [MF 8607]

In the boundary layer theory the Navier-Stokes equations for viscous flow in two dimensions, being themselves intractible, are simplified for high Reynolds numbers by neglecting certain terms. The chief simplification arises from the elimination of the pressure, by assuming that its tangential derivative is independent of the depth in the layer and hence is equal to the corresponding derivative determined by the main flow (potential flow) outside the layer. Although it would seem to be a purely mathematical problem to justify this assumption, no proof has been forthcoming except in special cases, and it is customary to appeal to physical intuition. In the present paper an attempt is made to formulate (but not to prove) the basic assumptions underlying the boundary layer theory in a precise form as weak as possible. The resulting postulates, albeit very complicated, are shown to be sufficient for the above mentioned simplification. They are apparently weaker than the usual postulates, for they do not explicitly imply that the pressure is constant across the layer, as is ordinarily assumed.

P. W. Ketchum (Urbana, Ill.).

Prandtl, L. Bemerkungen zur Theorie der freien Turbulenz. Z. Angew. Math. Mech. **22**, 241–243 (1942). [MF 8907]

As an introduction to the paper of H. Görtler reviewed below, Prandtl discusses the concept of the mixing length, formulated by him in 1925, and its application to the problems of "free turbulence." In this theory the exchange coefficient ϵ, which appears in the expression for shear stress, is put equal to $l^2|\partial\bar{u}/\partial y|$, where $\bar{u}$ is the mean velocity in the x direction and l is the mixing length. The author states that this theory, with l taken as a function of position, has led to pointed velocity profiles in contradiction with experimental results. New experimental results of Reichardt [V.D.I. Forsch.-Heft 414, Berlin, 1942] in cases of free turbulence have suggested instead the expression

$$\epsilon=\kappa b(\bar{u}_{\max}-\bar{u}_{\min}),$$

where κ is a constant, b is the breadth of the mixing zone at a certain cross-section and $\bar{u}_{\max}$, $\bar{u}_{\min}$ are the maximum and minimum values of $\bar{u}$ at that section. The author criticizes certain theoretical speculations made by Reichardt [loc. cit.]. In particular, one of Reichardt's equations is not invariant under the superposition of a constant velocity in the y direction, and therefore violates the principle of Newtonian relativity. *W. R. Sears* (Inglewood, Calif.).

Görtler, H. Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Näherungsansatzes. Z. Angew. Math. Mech. **22**, 244–253 (1942). [MF 8908]

Certain problems of "free turbulence" are treated according to Prandtl's new assumption regarding the exchange coefficient [see the preceding review]. The first is the plane problem of the mixing of two parallel streams of different mean speeds. Here Prandtl's rule leads to ϵ proportional to x, the distance downstream. The stream function has the form $\psi=xUF(y/x)$, and the approximate equation of motion, assuming constant pressure, takes the form

$$F'''+2\sigma^2FF''=0,$$

where U and σ are constants. This equation is solved by an iteration process and numerical results are given. A certain indeterminacy of the solution, characteristic of the boundary-layer equations, is discussed. The calculated velocity profiles are in good agreement with measurements of Reichardt [V.D.I. Forsch.-Heft 414, Berlin, 1942].

The second problem is that of the spreading of a plane jet. In this case Prandtl's assumption leads to ϵ proportional to $\sqrt{x}$. The approximate equation of motion is solved in terms of hyperbolic functions, and again the numerical results are in good agreement with Reichardt's measurements, except near the outer edges of the mixing region. A comparison with the results of Tollmien, based on Prandtl's earlier assumptions regarding ϵ, shows a more rounded profile near the center of the jet according to the new theory. An examination of the validity of the assumption of constant pressure is carried out, following Tollmien. It is found that the maximum pressure differences in both cases considered above are extremely small.

Finally the author treats the case of the plane wake at a large distance downstream of a cylinder. In this case, at least in the first approximation, ϵ is constant. The equations of motion are linearized under the assumption that $u=U_0+u_1$, where $u_1\ll U_0$ and also $v\ll U_0$. The solution is easily found in terms of the error function. The theoretical velocity profiles are in good agreement with the corresponding experimental results. *W. R. Sears.*

Kalikhman, L. E. A new method for calculating the turbulent boundary layer and determining the separation point. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 165–169 (1943). [MF 8683]

The equation of the plane stationary boundary layer in incompressible flow is written in the form

$$\frac{1}{\bar{U}R_*}\frac{dR_\vartheta}{d\bar{s}}-R_\vartheta\omega^2(H+1)=1/\zeta^2, \tag{1}$$

where

$$R_\vartheta=\bar{U}\bar{\vartheta}R_*,\quad \omega^2=-(1/\bar{U}^2R_*)d\bar{U}/d\bar{s},\quad \zeta=(\rho U^2/\tau_0)^{\frac{1}{2}},$$

$$R_*=Vb/\nu,\quad \bar{U}=U/V,\quad \bar{\vartheta}=\vartheta/b,\quad \bar{s}=s/b,$$

V is the velocity of the undisturbed flow, U the velocity outside the boundary layer, s the length of arc along the profile, b the chord of the profile, ρ the density, ν the kinematic viscosity, ϑ the momentum thickness, H the displacement thickness and τ_0 the shear stress at the wall. The von Kármán logarithmic resistance law is introduced in the form suggested by Squire and Young: $R_\vartheta=\text{const.}\ e^{\varkappa\zeta}$; the substitution $z=R_\vartheta\zeta^2$ then reduces (1) to a first-order differential equation for z. The author writes $K=1+2/\varkappa\zeta$ and assumes that

$$K\cdot(H+1)=\text{const.}\cong 3$$

and that the mean value of K is 1.2. The differential equation can then be integrated; the result is

$$z=\bar{U}^{-3}\left(C+1.2R_*\int_{\bar{s}_t}^{\bar{s}}\bar{U}^4d\bar{s}\right),$$

where $\bar{s}_t$ is the value of $\bar{s}$ at the beginning of the turbulent region and C is a constant. This result permits numerical calculation of the boundary layer and shearing stress of an airfoil profile, for example. The author outlines the detailed procedure to be followed. He states that, although the pressure-gradient actually affects H and the resistance law, the resultant effect on R_ϑ is small, and that good accuracy is obtained in the results even at critical angles of incidence.

The behavior of the shear stress τ near the wall at the point of separation can be deduced from the boundary-layer equations. Combined with Prandtl's equation relating τ and the velocity gradient normal to the surface, this leads to an expression for the velocity distribution in the laminar sublayer. By assuming the Reynolds number of the laminar sublayer to have a known value, the author obtains a rather complicated equation expressing the criterion for separation. He finally states that, within the limits of such values of ω as are of practical interest, the criterion can be represented by

$$L=R_\vartheta\omega^2R_\vartheta^{-0.08}=\text{const.}$$

and that the value of the constant, according to available experiments, is 0.0013. Some experimental results regarding L and the position of the separation point are shown for two airfoil profiles. *W. R. Sears* (Inglewood, Calif.).

Rhodes, L. S. The relationship between Reynolds number and velocity distribution. J. Appl. Mech. **10**, A-21–A-22 (1943). [MF 8044]

The note deals with a turbulent flow in a circular pipe of radius R. It is assumed that

(a) $u(y)/u(R)=(y/R)^n$, (b) $\log f=mN_R+\text{const.}$,

and (c) the shearing stress at the pipe wall does not depend upon the radius of the pipe. Here $u(y)$ is the (mean) velocity

at a distance y from the wall, N_R the Reynolds number, f the friction factor and n, m constants. Assumption (b) is justified by the fact that the slope m of the curve obtained by plotting $\log f$ against N_R (Stanton diagram) changes very slowly. By elementary considerations the author shows that $n=-m/(2+m)$ and that the average kinetic energy (of the mean motion) per unit weight of the fluid is $kV^2/2g$, where V is the average velocity and

$$k=(n+1)^2(n+2)^2/4(3n+1)(3n+2).$$

L. Bers (Providence, R. I.).

Reichardt, H. Die Wärmeübertragung in turbulenten Reibungsschichten. Z. Angew. Math. Mech. **20**, 297–328 (1940). [MF 8667]

The author first reviews briefly earlier theories of heat transfer in turbulent flows (for example, Prandtl, von Kármán, Taylor, Mattioli and Hofmann). He then sets up a formula of general validity for the calculation of the temperature as a function of the velocity by integration through the boundary layer. The integral involves the variation, with velocity, of the ratio τ_t/τ_m of turbulent to molecular shear stress and the "generalized Prandtl number" Pr', which is defined as the product of the Prandtl number Pr and the ratio of the turbulent exchange coefficients for heat and momentum. The domain of integration is divided into three regimes: (1) the laminar sublayer, (2) the transition layer, and (3) the turbulent layer. In the first and third of these certain simplifications of the integral can be made. The transition layer has been investigated experimentally by the author and his colleagues; their results indicate that the ratio $\tau_t/(\tau_m+\tau_t)$ is a linear function of the velocity. Numerical calculations are carried out for the temperature distribution near a smooth plane wall and for the heat transport in channels and pipes, as functions of the Prandtl and Reynolds numbers. In all these cases it is assumed that the fluid properties (viscosity and heat conductivity) are constant. A brief investigation indicates that variation of these properties with temperature will not greatly affect the velocity at the outer edge of the laminar layer although it will affect its thickness. Finally, the heat-transfer coefficient is defined in terms of the maximum temperature difference, and it is evaluated according to the present theory. Its numerical values are compared with measurements at both high and low Prandtl numbers, and good agreement is found. *W. R. Sears.*

Reichardt, H. Heat transfer through turbulent friction layers. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1047, 58 pp. (1943). (5 figures) [MF 9069]

Translation of the paper reviewed above.

Wieghardt, K. Berichtigungen zu: Zur Theorie der Wirbelbewegung. Z. Angew. Math. Mech. **22**, 176 (1942). [MF 8906]

This paper appeared in the same Z. **22**, 58–60 (1942); these Rev. **4**, 175.

Malurkar, S. L. Dynamics of thunderstorms. Proc. Indian Acad. Sci., Sect. A. **18**, 20–27 (1943). [MF 9063]

It is known that, owing to the effect of viscosity, superadiabatic lapse rates may occur only over small height intervals in the atmosphere. The superposition of potentially colder and heavier over potentially warmer and lighter air cannot extend through large vertical distances, so that this factor cannot play an important role in the development of thunderstorms. If moister air is situated underneath drier air, instability would also exist. However, the author shows that instability produced in this manner could also extend only through a thin layer. Hence it is concluded that other causes must be responsible for the initial convection in thunderstorms, such as horizontal temperature differences caused by unequal heating of the earth's surface, horizontal humidity or wind differences. It is pointed out that convective ascent due to unequal heating will not stop once the air parcel has reached its level of static equilibrium where its temperature becomes equal to that of the environment. The ascent will continue beyond this level owing to the upward momentum which the parcel of air has acquired. It is estimated that the extent of the overshooting is of the same order of magnitude as the distance from the starting point to the level of static equilibrium. This overshooting may often bring the air parcel to the condensation level in cases where the customary treatment of adiabatic ascent would seem to indicate that the parcel comes to rest considerably below this level. By means of these dynamic considerations which take into account the overshooting of the parcel air, it is possible to account for a greater number of thunderstorms than by means of quasi-static considerations, and thunderstorms may be more violent than the static treatment would indicate. *B. Haurwitz* (Cambridge, Mass.).

Leibenson, L. S. On the question of the distribution of wind with height. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1943**, 42–52 (1943). (Russian. English summary) [MF 8500]

Considered is the question of the distribution of wind with height. As the height where gradient wind settles is comparatively not very large, the change of the density of atmosphere with the height is neglected. The study of the problem is based on the well-known equations of Ackerblom. Turbulent viscosity is supposed to change according to the power law. On purpose to take into consideration the influence of the roughness of the earth's surface it is made use of the method being applied for the study of the influence of roughness of the bottom on the current of streams. From the investigation follows that the height at which gradient wind settles decreases with increasing roughness of the earth's surface. For the approximate integration of the equation of Ackerblom it was made use of the form of the integral condition given by the author in the theory of the boundary layer. *Author's summary.*

Berson, I. S. The hodographs of reiterated-reflected, reflected-refracted and refracted-reflected waves. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 275–289 (1942). (Russian. English summary) [MF 8524]

The author derives the equations for hodographs of longitudinal waves which are repeatedly reflected or refracted and reflected. He investigates the possibility of reducing such hodographs to hodographs of waves which are reflected only once. Some characteristic geometrical properties of these hodographs are discussed. The paper is of interest to the geophysicist rather than to the mathematician. *S. Bergman* (Providence, R. I.).

Theory of Elasticity

Lotze, Alfred. Die Vektorgleichungen für elastische Medien. Deutsche Math. 6, 352–370 (1942). [MF 8611]

In this paper the author applies the notation and methods of Grassmann to the study of the elastic deformation of bodies. Such well-known results as (1) the principal directions of a symmetric tensor (corresponding to distinct roots of the characteristic equation) are orthogonal; (2) the derivation of the dynamical equations for elastic media, are given. *N. Coburn* (Austin, Tex.).

Bers, Lipman and Gelbart, Abe. On a class of differential equations in mechanics of continua. Quart. Appl. Math. 1, 168–188 (1943). [MF 8806]

This paper is concerned with the equations (*) $u_x=\tau_1(y)v_y$, $u_y=-\tau_2(y)v_x$ which are frequently found in applied mathematics. A set of particular solutions can be found by putting

$$U=\int_{z_0}^{z} u\,dx-\tau_2 v\,dy, \quad V=\int_{z_0}^{z} v\,dx+(u/\tau_1)dy, \quad z=x+iy,$$

and by repeating this process indefinitely.

The authors call $f=u+iv$ and $F=U+iV$ a Σ-monogenic function and its Σ-integral (relative to the coefficients τ_1, τ_2), respectively. The Σ-differentiation is the inverse process to the Σ-integration. Starting with $f=1$ and $f=i$, the authors introduce by Σ-integration the formal powers $(\alpha+\beta i)Z^{(n)}(z)$. It can be shown that any Σ-function can be represented around the origin by a formal power series $\sum a_n Z^{(n)}$. The correspondence between this series and $\sum a_n z^n$ facilitates the construction of particular solutions of (*), for instance, of Σ-exponential and trigonometrical functions, which are also obtainable by the method of separation of variables. Special attention is given to the case $\tau_1=\tau_2=y^{-p}$. In the physical examples (flow of an incompressible fluid with rotational symmetry, torsion of bodies of revolution, two-dimensional potential gas flow) the main purpose of the authors was to present known facts from a more unified viewpoint. The authors mentioned that for the gas dynamical equations the formal powers have been considered independently by S. Bergman and A. Vaszonyi. The reviewer remarks that, in the case $\tau_1=\tau_2=y^{-1}$, a process identical with the repeated Σ-integration and differentiation was introduced by Beltrami in two papers published in 1878 and 1880 [Opere Matematiche, vol. 3, U. Hoelpli, Milano, 1920, pp. 115–128, 349–382], where several particular solutions are given and applied to problems of electrostatics. Beltrami's ideas were used later in several problems [see, for instance, R. Serini, Deformazioni simmetriche dei corpi elastici, Accad. Naz. Lincei. Rend. (5) 28, 343–347 (1919)]. *A. Weinstein.*

Lourye, A. I. Contact problems of the theory of elasticity. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] 5, 383–408 (1941). (Russian. English summary) [MF 7722]

The paper deals with the state of stress produced in an elastic half-space by impressing a rigid stamp. By means of the general solution of the differential equations of the theory of elasticity due to P. F. Papkovitch [Theory of Elasticity (Russian), 1939] and H. Neuber [Z. Angew. Math. Mech. 14, 203–212 (1934)] this mechanical problem is reduced to the following problem in potential theory: to determine the density of a simple layer spread over a given domain Ω of the x,y-plane and producing a potential which coincides on Ω with a given function $\psi(x,y)$. The domain and the function ψ are determined by the shape of the stamp and by the components of the infinitesimal displacement given to the stamp which initially had but one point of contact with the half-space. The density of the layer represents, but for a constant factor, the normal stress distribution along the surface in contact. Unless edges of the stamp enter into the elastic domain, the density must vanish on the boundary of Ω. The author discusses the cases where Ω is a circle or an ellipse. In the first case he introduces curvilinear coordinates s, μ, φ given by

$$x=a\sqrt{1+s^2}\sqrt{1-\mu^2}\cos\varphi, \quad y=a\sqrt{1+s^2}\sqrt{1-\mu^2}\sin\varphi, \quad z=as\mu.$$

The above potential theoretical problem is first solved for an oblate ellipsoid of revolution $s=s_0$, by developing the potential and the density in a series of certain spherical harmonics. Similarly, in the case where Ω is an ellipse general elliptic coordinates and Lamé's functions are used.

The formulae for a circular Ω are applied to the case of an excentrically loaded plane elliptic stamp, previously treated by V. M. Abramov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 23, 759–764(1939); these Rev. 1, 288] and to the case of a rotationally symmetric stamp bounded by the surface $\zeta=A\rho^s$, $s\geqq 1$. The case $s=1$ (conical stamp) has been treated by A. E. H. Love [Quart. J. Math., Oxford Ser. 10, 161–175 (1939); these Rev. 1, 91]. An elliptical Ω leads to the excentrically loaded plane elliptical stamp, already discussed by the author in a previous note [C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 106–109 (1940); these Rev. 2, 175], and to the case of a stamp bounded by an elliptical paraboloid. This last case is equivalent to Hertz's contact problem. The author emphasizes the difference between his "direct" method and the inverse method used by Hertz and subsequent writers. *L. Bers* (Providence, R. I.).

Steuermann, E. J. On one generalization of Hertz' problem. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] 5, 409–418 (1941). (Russian. English summary) [MF 7723]

This paper deals with local deformations obtained by compressing two elastic bodies. It is assumed that the initial distance between the two surfaces is given by $d=A(x^2+ky^2)^2+$terms of higher order, a case not contained in Hertz's classical solution. The author first computes the potential v_m of a simple layer of density

$$\mu(x', y')=(1-x'^2/a^2-y'^2/b^2)^{m+\frac{1}{2}}$$

spread over the elliptic disk $x^2/a^2+y^2/b^2<1$. It turns out that, for points of the disk, v_m is a polynomial of degree $2m+2$ in x and y. Next the solution of the contact problem is assumed to be of the following form: the surface of contact is bounded by an ellipse of semi-axes a, b; the stress-distribution is given by $p(x', y')=C(v_0-\frac{3}{4}v_1)$. The author shows that, for given values of A and k, a, b and C can be determined in such a way that this assumption actually presents a close approximation to the solution of the above contact problem. *L. Bers* (Providence, R. I.).

Ghosh, S. On plane strain and plane stress in aelotropic bodies. Bull. Calcutta Math. Soc. 34, 157–169 (1942). [MF 8956]

The first part is concerned with the conditions for the existence of a state of plane strain in an aeolotropic body. The most important cases of aelotropic bodies in which these conditions are satisfied are: (a) materials which possess an axis of symmetry perpendicular to the plane of strain, (b) bodies with three perpendicular planes of symmetry,

one of which is parallel to the plane of strain. The second part investigates the conditions for the existence of a state of plane stress in the case of materials of the type defined under (b). Unless the elastic constants of the material satisfy a certain equation, these conditions restrict severely the possible types of plane stress: if x,y are rectangular coordinates in the plane of stress, the stress function is a polynomial in x,y of a degree not higher than six. *W. Prager.*

Usunoff, Nicola. Über den Spannungszustand im Halbraum bei halbkugelförmiger Druckverteilung. Z. Angew. Math. Mech. **22**, 262–269 (1942). [MF 8910]

The author considers the stresses in the semispace $x>0$ due to a semispherical distribution of normal stresses acting on the plane $x=0$. Boussinesq [B. A. und L. Föppl, Drang und Zwang, Bd. 2, Oldenbourg, München, 1920, p. 226] has computed the stress components at any point in such a solid due to a concentrated force acting normal to $x=0$. By use of Boussinesq's results, the author obtains expressions for the stress components σ_x, τ in terms of a double integral. Next, the author evaluates this integral by use of elliptic functions. Stress curves are plotted for various values of σ_x, τ. *N. Coburn* (Austin, Tex.).

Green, A. E. A note on stress systems in aeolotropic materials. I. Philos. Mag. (7) **34**, 416–420 (1943). [MF 8736]

Green, A. E. A note on stress systems in aeolotropic materials. II. Philos. Mag. (7) **34**, 420–422 (1943). [MF 8737]

The first paper extends previous treatments of generalized plane stress analysis in homogeneous aeolotropic materials [Green and Taylor, Proc. Roy. Soc. London. Ser. A. **173**, 162–172 (1939); Green, ibid., **173**, 173–192 (1939); **180**, 173–208 (1942); these Rev. **4**, 123] to the general case when the elastic properties of the material are specified by five independent constants. A general expression for the stress function is given and is specialized in the cases of an isolated force acting in the plane of an infinite sheet or at a point in the straight boundary of a semi-infinite sheet, and in a consideration of stress distributions in infinite sheets with circular holes. The theory presented is applied in the second paper to an extension of related work by Sen [Philos. Mag. (7) **27**, 596–604 (1939)]. *F. B. Hildebrand.*

Higgins, Thomas James. The approximate mathematical methods of applied physics as exemplified by application to Saint-Venant's torsion problem. J. Appl. Phys. **14**, 469–480 (1943).

A great number of approximation methods of applied mathematics are illustrated by discussing their application to the problem of torsion of prismatic elastic bars. A valuable list of references containing 72 numbers is given. *W. Prager* (Providence, R. I.).

Arutinyan, N. C. Approximate solution of the problem of torsion of bars having a polygonal cross section. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 19–30 (1942). (Russian. English summary) [MF 7732]

The torsion problem for elastic prismatic bars consists of integrating the equation

$$\Delta\phi(x, y) = -2 \tag{1}$$

under the boundary condition $\phi=0$ at the boundary of the cross-section. The author uses nonorthogonal curvilinear coordinates ρ, φ given by

$$x=a(\sqrt{\rho}\cos\varphi)^{1/n}, \quad y=b(\sqrt{\rho}\sin\varphi)^{1/n},$$

where $a>0$, $b>0$ and n is an integer. In these coordinates (for an n of the order of magnitude 15–25) the boundaries of several practically important polygonal cross-sections with rounded off corners can be represented with a great degree of accuracy by equations of the form $\rho=\rho_1=\text{const.}$, $\rho=\rho_2=\text{const.}$, $\varphi=0$, $\varphi=\alpha\pi/2$.

The solution of the boundary value problem is carried out by an approximate method due to Kantarovitch [Kantarovitch, L. V. and Krylov, A. N., Approximate solution of partial differential equations (Russian), 1939]. The solution is assumed to be of the form $\phi=\sum_1^m f_k(\rho)\omega_k(\varphi)$, where the functions $\omega_k(\varphi)$ are fixed and such that the boundary conditions are satisfied, and the functions $f_k(\rho)$ are to be determined. (The practical success of the methods obviously depends upon the choice of the functions ω_k.) This expression for ϕ is then introduced into the variational problem for which (1) is the Euler-Lagrange equation. The Euler-Lagrange equations now become ordinary linear differential equations. The substitution $\rho=e^t$ leads to a nonhomogeneous system of ordinary differential equations with constant coefficients. Two cases are discussed in detail: the thin-walled angle-iron and the thin-walled U-iron. In the first case the author sets $\omega_k=\sin k\varphi\cos k\varphi$, in the second $\omega_k=\sin k\varphi$. Only the first approximation ($m=1$) is taken into account. The computations check well with known theoretical, semi-theoretical and experimental results. *L. Bers* (Providence, R. I.).

Pojalostin, A. I. and Riz, P. M. On oblique bending in the non-linear theory of elasticity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 375–380 (1942). (Russian. English summary) [MF 7588]

This paper is considering deviations from superposition principles when the terms of the second power in the equations of the theory of elasticity are taken into consideration. It is shown that torsion arises due to the simultaneous action of two bending moments in the main planes. This additional torsion is not considered in the elementary theory of oblique bending and may be highly significant for bars of thin cross section. *Author's summary.*

Shapiro, G. S. Design of a plate conceived as an infinite band resting upon elastic foundation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 202–204 (1942). [MF 8661]

In this note the author gives a resume of results obtained for the elastic deflection function $w(x, y)$ of a plate in the shape of an infinite band of width $2b$ and rigidity $D=Eh^3/12(1-\sigma^2)$. A concentrated force P is applied normal to the plate at the point $(0, c)$ and it is assumed that the reaction of the foundation is proportional to the deflection w. The differential equation to be studied is of the biharmonic type, namely

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{k}{D}w=\frac{p}{D},$$

where k is the coefficient of the subgrade, p is the normal load per unit area of the plate (expressed in terms of P by the interrupter of N. M. Gersevanov). Two types of boundary conditions are studied. In each case the solution of the above equation is expressed in terms of integrals of the Fourier type. *N. Coburn* (Austin, Tex.).

Girkmann, K. Angriff von Einzellasten in der streifenförmigen Scheibe. Ing.-Arch. **13**, 273–284 (1943). [MF 9071]

The author investigates the state of stress of a plate, infinite in one direction and under the action of various types of concentrated loads which are parallel and perpendicular to the free edges of the plate. This is accomplished by writing the Airy stress function in Fourier integral form in order to obtain the singular part of the solution. A particular solution which satisfies the plate equation and the conditions on the free boundary is then constructed with the aid of the Fourier integral theorem in order to supply the regular part of the solution.

A. E. Heins (Cambridge, Mass.).

Goland, Martin. The influence of the shape and rigidity of an elastic inclusion on the transverse flexure of thin plates. J. Appl. Mech. **10**, A-69–A-75 (1943). [MF 8425]

The types of inclusions considered are elastic circular, rigid circular and rigid elliptic. For each of these types the influence of the inclusion on the stress distribution is studied for the fundamental cases of loading that, in a homogeneous plate, would result in plain bending, cylindrical bending, uniform twist or uniform shear. *H. W. March.*

Müller, Wilhelm. Die Durchbiegund einer Kreisplatte unter exzentrisch angeordneten Lasten. Ing.-Arch. **13**, 355–376 (1943). [MF 9074]

The author derives Michell's solution for the deflection of an elastic clamped-edge circular plate with excentric point load by using bipolar coordinates, as has previously been done by E. Melan [Eisenbau **11**, 190 (1920)] and W. Flügge [Berechnung von Kreisplatten unter Einzellasten, Springer, Berlin, 1929]. He then finds by summation the solution for a line load distributed uniformly over an excentric circle. Here reference should be made to previous solutions of the same problem by H. Schmidt [Ing.-Arch. **1**, 147 (1930)] and S. Serschgorin [Appl. Math. Mech. **1**, 159 (1933)]. The paper contains some results of numerical evaluations of the solutions which appear not to have been published previously. For the solution in closed form of the problem of the plate with elastically supported edge, for which the bipolar coordinate method appears not to be suitable, see E. Reissner [Math. Ann. **111**, 777–780 (1935); Z. Angew. Math. Mech. **17**, 57–58 (1937)]. *E. Reissner.*

Müller, Wilh. Zur Biegungstheorie einer Rohrflanschverbindung. Ing.-Arch. **13**, 185–197 (1942). [MF 8591]

A cylindrical tube with terminal flanges is under the action of axial compressive forces, internal pressure and bending moments applied around the circular periphery at the ends. Approximate analysis of the state of stress is developed under several simplifying assumptions which permit the application of the theory of axially symmetrical states of stress and the theory of thin plates and shells.

H. W. March (Madison, Wis.).

Pflüger, A. Zur Stabilität der dünnen Kegelschale. Ing.-Arch. **13**, 59–72 (1942). [MF 8577]

In the first part of this paper the author supplements an earlier article [Ing.-Arch. **8**, 151–172 (1937)] with numerical results for the buckling loads of thin conical shells of linearly varying thickness under axial and external pressure loads. This critical load is found from the vanishing of a determinant which arises from satisfying the differential equations for the displacements.

In the second part, the critical twisting moment for the torsional buckling of truncated conical shells is investigated when rather extensive approximations and simplifying assumptions are employed. In this case the shells are of constant thickness and the Ritz method of minimizing the energy is used to obtain the characteristic determinant for the buckling moment. The approximation functions for the displacements are analagous with and degenerate to those functions which solve the exact problem in the torsion of thin cylindrical shells. For conical shells of small taper, the results are expressed by an empirical formula in terms of the critical twisting moment of a cylindrical shell and the taper angle. *D. L. Holl* (Ames, Iowa).

Galerkin, B. Equilibrium in an elastic spherical shell. Appl. Math. Mech. [Akad. Nauk SSSR. Pri[illegible] Mat. Mech.] **6**, 487–496 (1942). (Russian. English summary) [MF 8363]

This paper is concerned with the determination of stresses and deformations in a spherical shell subjected to a distribution of forces on the surface of the shell. The method of solution consists of introducing three independent stress functions, with the aid of which one can calculate stresses and displacements satisfying the conditions of equilibrium and Beltrami's compatibility equations. The following cases are considered: (a) a shell formed by two concentric spheres and a cone whose vertex is at the center of the sphere (this includes the case of a complete spherical shell); (b) an open spherical shell bounded by two concentric spheres and two coaxial cones whose vertices are at the center of the sphere. Solutions are given in the form of infinite series involving Legendre's functions. *I. S. Sokolnikoff* (Madison, Wis.).

Wegner, Udo. Zur Stabilität des mehrfeldrigen elastisch gestützten Stabes. I. Luftfahrtforschung **19**, 374–380 (1943). [MF 8927]

A new method is developed for determining the resistance to buckling of a column of the type described in the title. The conditions of equilibrium at the points of support lead to a system of linear difference equations subject to conditions of continuity at the supports. With the use of matrices a transcendental equation is derived which expresses the condition for buckling. *H. W. March* (Madison, Wis.).

Schunck, T. E. Die Berechnung schwach gekrümmter S-förmiger Stäbe. Ing.-Arch. **13**, 73–79 (1942). [MF 8578]

The bars under consideration are made up of two thin semi-circular bars of different radii. Tensile or compressive loads act along a line through the centers of the semi-circular bars. The ends to which the loads are applied are taken to be either hinged or clamped. Expressions are obtained for the relative displacement of the ends under a given load and for the relative displacement associated with a given permissible maximum stress. *H. W. March.*

Ylinen, Arvo. Die Differentialgleichung der Biegungsschwingungen eines axial belasteten geraden Stabes, dessen Material dem Hookeschen Gesetz nicht folgt. Z. Angew. Math. Mech. **22**, 163–164 (1942). [MF 8898]

This paper derives the differential equation governing small bending vibrations of an axially loaded straight rod, the material of which follows a nonlinear stress-strain law,

in the form

$$\rho F\frac{\partial^2 y}{\partial t^2}-P\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2}{\partial x^2}\left[c_1\frac{\partial^2 y}{\partial x^2}-c_2\left(\frac{\partial^2 y}{\partial x^2}\right)^2+c_3\left(\frac{\partial^2 y}{\partial x^2}\right)^3-\cdots\right]$$
$$=\rho\frac{\partial}{\partial x}\left(I\frac{\partial^2 y}{\partial t^2\partial x}\right)+Y,$$

where x and y are axial and transverse variables, ρ is the density, F the cross-sectional area, P the axial force, Y the transverse load and I the principal moment of inertia of a section. The coefficients $c_i(x)$ are defined in terms of the higher moments of a section and in terms of certain values of the higher derivatives of the stress relative to the strain [Ylinen, Ann. Acad. Sci. Fennicae Ser. A. **57** (1941)]. If Hooke's law applies there follows $c_1=EI$, $c_2=c_3=\cdots=0$, and the known equation obtains. *F. B. Hildebrand.*

Schallenkamp, A. Transversalschwingungen eines einseitig eingespannten Trägers bei bewegter Last. Ing.-Arch. **13**, 267–272 (1943). [MF 9070]

The paper deals with the transverse oscillations of a uniform beam of length l fastened at one end and carrying a moving load. This problem has an application to the vibration of a cannon barrel as the projectile passes through it. A general solution is difficult because the partial differential equation has variable coefficients. In this paper it is assumed that the fixed end at $x=0$ is not rigid but has an elastic resistance to variation of slope. The author restricts the complexity of the problem by assuming that the deformation $y(x, t)$ at any time t is parabolic, so that

$$y(x, t)=q_1(t)y_1(x)+q_2(x)y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are polynomials of degree 1 and 2, respectively, which for convenience are normalized and orthogonal with respect to the interval from 0 to l. From the energy of the system he sets up the Lagrangean equations for the determination of the parameters q_1 and q_2 and, first of all neglecting the inertia of the moving load, obtains solutions for the cases where the moving load has (1) constant velocity and (2) constant acceleration. Next, to take account of inertia, he represents the path of the load by a few terms of the expansion in Legendre polynomials with undertermined coefficients, which are then determined by methods based on the cases first considered. The case where the beam is initially curved is similarly treated.

W. E. Milne (Corvallis, Ore.).

Woinowsky-Krieger, S. Kippschwingungen und dynamische Kippstabilität der I-Träger bei Belastung durch Endmomente. Ing.-Arch. **13**, 197–210 (1942). [MF 8592]

An I beam is subjected to bending moments applied at its ends. An axial compressive force may or may not be present. It is shown that such a beam is capable of executing modes of vibration in which there is coupling between bending and twisting modes. The connection with the usual theory of stability is exhibited by the fact that, under constant applied bending moment and axial load, for each mode of vibration having nodes equally spaced longitudinally there are two natural frequencies, and that one of these frequencies vanishes when the applied moment reaches the critical value that causes lateral buckling of the beam. For forced vibrations associated with harmonic applied moments the analysis leads to a fourth order linear differential equation with periodic coefficients. From the form of the solution of this equation it is possible to predict resonance phenomena when the frequency of the forced vibrations lies within certain limits. These limits are determined for a few special cases. *H. W. March.*

Woinowsky-Krieger, S. Über die Biegeschwingungen eines Kreisringes unter gleichmässig verteiltem pulsierendem radialem Druck. Ing.-Arch. **13**, 90–96 (1942). [MF 8580]

The author studies the dynamic stability of a circular ring under the influence of pulsating radial pressure which is uniformly distributed along the circumference of the ring. From the mathematical point of view the problem is intimately related to that of the dynamic stability of a straight rod under a pulsating axial force; both problems lead to Mathieu's equation. From the mechanical point of view, however, the present problem is more complex insofar as flexural vibrations not only in the plane of the ring but also normal to this plane must be taken into consideration. In the case of the numerical example given in the paper the ring has equal bending stiffness in and normal to its plane; the regions of instability corresponding to the two types of flexural vibration coincide practically. It is not discussed whether this will always be the case when the two bending stiffnesses are equal. *W. Prager* (Providence, R. I.).

Mettler, E. Über die Stabilität erzwungener Schwingungen elastischer Körper. Ing.-Arch. **13**, 97–103 (1942). [MF 8581]

It is a well-known fact that the stability of the equilibrium of an elastic system can be discussed in a satisfactory manner only if the nonlinear terms in the expressions for the strain components are taken into consideration. It is pointed out in the present paper that a similar situation exists as far as the stability of the forced vibrations of an elastic system is concerned. The vibrations of a straight rod under the influence of a pulsating axial load are discussed as examples. The differential equation on which the discussion of this problem is usually based is seen to be a rather simplified form of the complete differential equation furnished by Hamilton's principle when the higher order terms are retained in the expression of the strain energy. *W. Prager.*

Klöppel, K. und Lie, K. H. Lotrechte Schwingungen von Hängebrücken. Ing.-Arch. **13**, 211–266 (1942). [MF 8593]

Vertical vibrations of suspension bridges are investigated on the basis of a theory of second order [K. H. Lie, Stahlbau **14**, 65 (1941)]. The suspension cable is assumed either anchored to the ground or to the stiffening truss. The latter is either of a single span, of three separate spans or a continuous truss. Free vibrations under the following assumptions are studied: (1) uniformly distributed mass along the truss, (2) additional concentrated mass, (3) a mass uniformly distributed over a part of the truss. The damping is first neglected; the amplitude of free vibrations of the truss satisfies an integro-differential equation whose eigenvalues give the natural frequencies. The presence of a viscous damping decreases slightly these frequencies. Forced vibrations due to the following types of loads moving with a constant speed are studied: a constant and a time-periodic concentrated force, a constant force distributed uniformly over a certain length of the truss. The loads are expressed as Fourier series of a coordinate along the geometric axis of the truss. About a fourth of the paper is devoted to several numerical examples. *I. Opatowski* (Chicago, Ill.).

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